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THE EISENSTEIN CONSTANT

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Introduction. Let $f(X, Y) \in \mathbb{Z}[X, Y]$ be of degree n in the variable Y and have coefficients bounded by H. A remark of Eisenstein [Eis] points out that, if y is a formal series

$$y = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \cdots$$

which satisfies $f(X, y) = 0$, then there are natural numbers a_0 and a so that

$$a_0 a^i \alpha_i \qquad (i = 0, 1, \ldots)$$

are algebraic integers. (See [Die], p. 327 ff. for an old-fashioned proof.) We assert that the Eisenstein constant a is bounded by $c(n)H^{2n-1}$, where

$$c(n) = n^n(2n-1)!\,\mu_n\lambda_n^n,$$

$$\mu_n = \prod_{p \leqslant n} p \leqslant n!,$$

$$\lambda_n = \exp(\tau_n + \psi(n)),$$

$$\tau_n = \sum_{p \leqslant n} \frac{1}{p-1} \log p \leqslant \log 2 + 0.5 \log^2 n,$$

$$\psi(n) = \sum_{p \leqslant n} \left[\frac{\log n}{\log p}\right] \log p \leqslant 1.22n + 2.24 \log^2 n.$$

(For this estimate for ψ see [Sh, p. 389].) Thus,

$$c(n) \leqslant 4.8(8e^{-3} n^{4+2.74 \log n} e^{1.22n})^n.$$

The factor H^{2n-1} appears for the following reason: Suppose we write the discriminant $R(f, f_Y)$ of f in the form

$$D(X) = X^l(D_0 X^\mu + D_1 X^{\mu-1} + \cdots + D_\mu), \qquad D_\mu \neq 0, \quad D_j \in \mathbb{Z}.$$

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