EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

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13 Find the power series expansion of a second solution to the equation

$$x^2y'' + 3x'y' + y - xy = 0$$

from the class of November 8.

14 Let ρ be a maximal local exponent of $L \in \mathcal{O}[\partial]$, of multiplicity 2. Show that there exist $h_0, h_1 \in \mathcal{O}$ such that $y_1 = x^\rho h_0(x)$ and $y_2 = x^\rho h_1(x) + x^\rho \log(x) h_0(x)$ are solutions of $Ly = 0$. Determine the order of vanishing of h_0 and h_1 at 0.

$Hint.$ Use the description of the automorphism u in the Normal Form Theorem.

15 Let ρ be a maximal local exponent of an Euler operator $E \in \mathcal{O}[\partial]$, with multiplicity m, and let E act on $\mathcal{F} = x^\rho \mathcal{O}[z]$ via $\partial x = 1$ and $\partial z = x^{-1}$. Show that the image of E equals $x\mathcal{F}$. Then determine the image under E of $x^\rho \mathcal{O}[z]_{<m}$ (polynomials in z of degree $< m$).

16 Let $\rho, \sigma \in \mathbb{C}$ be simple (= multiplicity 1) local exponents of an Euler operator E, and assume that $\rho - \sigma \in \mathbb{N}_{>0}$.

(a) Determine $E(\mathcal{F})$ for $\mathcal{F} = x^\sigma \mathcal{O}$ and $\mathcal{F} = x^\sigma \mathcal{O}[z]$. Find out whether $E(\mathcal{F})$ is strictly included in $x\mathcal{F}$ and, if yes, determine the gaps.

(b) Determine a (non-trivial) function space $\mathcal{F}$ similar to the above form such that E maps $\mathcal{F}$ onto $x\mathcal{F}$.