EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

Herwig HAUSER

17 Determine the kernels and images of the three operators

$$L = x^2\partial^2 - x\partial - x^3,$$

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acting on $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$.

Remark. As you might expect, the answers are quite different, for different reasons. For more details, see [Gann-Hauser, JSC, p. 4 and p. 9], availaible on www.hh.hauser.cc.

18 Let $y_1 = x^\rho, ..., y_m = x^\rho \log(x)^{m-1}$ be the solutions of the Euler equation $L_0 y = 0$ with respect to the local exponent ρ of multiplicity m, and let $L \in \mathcal{O}[\partial]$ have initial form L_0. Assume that ρ is maximal with respect to $\mathbb{Z}$ and that 0 is a regular point of L. Then the solutions of $Ly = 0$ are of the form, for $1 \leq i \leq m$,

$$y_1(x) = x^\rho h_1(x),$$

$$y_2(x) = x^\rho [h_2(x) + h_1(x)\log(x)],$$

$$y_i(x) = x^\rho [h_i(x) + h_{i-1}(x)\log(x) + \ldots + h_1(x)\log(x)^{i-1}],$$

with $h_1, ..., h_m$ holomorphic functions in $\mathcal{O}$.

Hint. Use the description of the automorphism u of $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ inthe normal form theorem.

19 Let $E = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ be an Euler operator with indicial polynomial $\chi(t) = (t-1)(t-3)^2$ and local exponents $\rho = 3$ of multiplicity $m = 2$ and $\sigma = 1$ of multiplicity 1. Let it act on $x\mathcal{O}[z]_{<2}$. Then

$$\underline{E}(x^k z^i) = x^k[(k-1)(k-3)^2 z^i + (3k-5)(k-1) i z^{i-1} + (6k-14) i^{\underline{2}} z^{i-2} + 6 i^{\underline{3}} z^{i-3}].$$

The kernel is $\mathrm{Ker}(E) = \mathbb{C}x \oplus \mathbb{C}x^3 \oplus \mathbb{C}x^3 z$. Determine the image $\mathrm{Im}(E)$!

20 Let L be an operator with initial form $L_0 = x^2\partial^2 - x\partial$, indicial polynomial $\chi(t) = t(t-2)$ and local exponents $\rho = 2$ and $\sigma = 0$, both of multiplicity 1. Find a suitable function space $\mathcal{F} \subset x^\sigma \mathcal{O}[z]$ for which one may hope to get again a normal form theorem for the extension $\underline{L}$ of L, reducing it to $\underline{L}_0$ by means of an automorphism of $\mathcal{F}$.

Hint. A suitable $\mathcal{F}$ will lie in $x^\sigma \mathcal{O}[z]_{<2} = x^\sigma \mathcal{O} z$, where $2 = m_\sigma + m_\rho$.