EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

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25 Let $y(x)$ be a root of $P(x,y) = y^2 - xy + x^3 \in \mathbb{Q}(x)[y]$.

(a) Show that it is not étale algebraic at 0. What about the other points of $\mathbb{C}$?

(b) Decompose $y(x) = k(x) + x^e a(x)$ with a étale algebraic at 0 and $a(0) = 0$, for some polynomial k.

(c) Find an integer $d > 0$ such that the first 100 coefficients of $y(dx)$ are integral.

26 Determine a differential equation of minimal order of $y(x) = \sqrt{x}\log(x) + 1$ and find its solutions.

27 Prove that any algebraic series can be decomposed into $y(x) = k(x) + x^e a(x)$ where k is a polynomial and a is étale algebraic.

Hint. Use $e = \operatorname{ord} \partial_y P(x, y(x))$, where P is the minimal polynomial of $y(x)$.

28 Try to prove directly that, for any $m \in \mathbb{N}$, the series $\sqrt[m]{1+\ell x}$ has integral coefficients, for some suitable integer $\ell > 0$. Then determine the minimal ℓ which does the job.