EXERCISES FUCHSIAN DIFFERENTIAL EQUATIONS FALL 2022

Herwig HAUSER

29 Let $r(t) \in \mathbb{Q}(t)$ be a rational function and consider the sequence $(c_k)_{k\in\mathbb{N}}$, given by the linear recursion

$$c_{k+1} = r(k)c_k,$$

for $k \geq k_0$, and some initial values $c_0, ..., x_{k_0-1}$. Let $y(x) = \sum c_k x^k$ be the generating function of $(c_k)_{k\in\mathbb{N}}$.

(a) Find a linear differential equation for $y(x)$.

(b) Illustrate your findings in three interesting examples.

30 Let now $Ly = 0$ be a linear differential equation with coefficients in $\mathbb{Q}(x)$. Assume that $L = L_0 + L_1$, for Euler operators of shifts 0 and 1. Let $y(x) \in \mathbb{Q}[[x]]$ be a power series solution.

(a) Determine a linear recursion for the coefficient sequence of $y(x)$.

(b) Illustrate your findings in three interesting examples.

31 Let $\delta = x\partial$ and consider the Euler differential operator $L = \delta^2 - 3\delta - 10$.

(a) Compute the indicial polynomial of L and the expansion of L in terms of ∂.

(b) Reduce $Ly = 0$ modulo all primes p and determine the respective solutions.

(c) Try to find a basis of polynomial solutions.

32 Find two different proofs of the following theorem of Kronecker:

Theorem. *If a univariate polynomial $P \in \mathbb{Q}[x]$ factors linearly modulo p for almost all primes p, then it factors linearly as a polynomial in $\mathbb{Q}[x]$.*