FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART I: EXAMPLES

Let us present a bunch of differential equations with different types of singularities. Most of them are classical and have a geometric or number theoretic background.

The numeration follows the forthcoming Lecture Notes on Fuchsian Differential Equations. The write-up is (very) preliminary and sketchy, be cautious for errors or typos. Nothing will be proven yet. References will be added later on.

(B8 0) Differential equations with constant leading term and holomorphic coefficients have no singularities in $\mathbb{C}$. They might have singularities at ∞. [Try to find an example of such a singularity at ∞!] At all non-singular points, Cauchy's theorem applies, and not much more can be said.

(B9 1) The Euler equation $\sum_{i=0}^n c_i x^i y^{(i)} = 0$ is the prototype of an equation with a regular singularity at 0 (and at ∞). Indeed, the quotients $\frac{p_i(x)}{p_0(x)} = \frac{x^{n-i}}{x^n}$ equal x^{-i} and have all poles of exact order i at 0. All differential equations with at most two regular singularities, say, at 0 and ∞, are already Euler equations. If all exponents ρ_i are distinct, the monomials $y(x) = x^{\rho_i}$ form a C-basis of solutions. If a local exponent ρ has multiplicity m, the respective solutions are x^ρ, $x^\rho \log(x)$,..., $x^\rho \log(x)^{m-1}$.

(B10 2) The (second order) hypergeometric equation was considered already by Euler (1707-1783) and studied later extensively by Gauss (1777-1855). It has the form

$$x(x-1)y'' + ((a+b+1)x - c)y' + aby = 0,$$

with $a, b, c \in \mathbb{C}$. At first glance, the equation may seem rather arbitrary. This is not the case: on the contrary! It has three singularities, namely at 0, 1 and ∞. All three are regular. All second order linear differential equations with three regular singularities are equivalent, via an automorphism of $\mathbb{P}^1_{\mathbb{C}}$, i.e., a Möbius transformation, to the above form: take a fractional linear transformation $x \to \frac{\alpha x+\beta}{\gamma x+\delta}$, with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and a multiplication of the y-variable by a monomial. So the Gauss hypergeometric equation covers all these cases. The exponents at 0, 1 and ∞ are 0 and $1-c$, respectively, 0 and $c-a-b$, respectively a and b. The position of the three singularities and the values of their exponents determine the hypergeometric equation completely.

A basis of solutions of the hypergeometric equation with parameters a, b, c [excluding some special values] is given by the hypergeometric series (denote by $a^{\bar{k}} = a(a+1)\cdots(a+k-1)$ the rising factorial or Pochhammer symbol)

$$y_1(x) = {}_2F_1(a,b;c;x) = \sum_{k=0}^\infty \frac{a^{\bar{k}} b^{\bar{k}}}{c^{\bar{k}} k!} x^k,$$

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$$y_2(x) = x^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; x).$$

Idea of proof. The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{P}^1_{\mathbb{C}}$ by Möbius transformation is 3-transitive: any triple of distinct points can be mapped by the action of $\mathrm{SL}_2(\mathbb{C})$ to any other triple of distinct points: If a_1, a_2, a_3 are three distinct points, apply $x \to \frac{(a_2-a_3)(x-a_1)}{(a_2-a_1)(x-a_3)}$ and obtain 0, 1, ∞ [Ince, p. 497]. Therefore, given a second order differential equation with three regular singular points, we may assume that these are located at 0, 1, and ∞. As the equation has order 2, there will be for each of these points 2 local exponents.

A prospective solution $y(x)$ of the differential equation is factored into $y(x) = x^{-s_0}(x-1)^{-s_1} \cdot z(x)$. This yields for $z(x)$ a new differential equation with the same singularities but modified exponents $(0, \rho_0 - \sigma_0)$, $(0, \rho_1 - \sigma_1)$ and $(\rho_\infty + \sigma_0 + \sigma_1, \sigma_\infty + \sigma_0 + \sigma_1)$. Thus the scheme of exponents has become the same as the one for the hypergeometric equation. As the location of the singularities and their local exponents determine the differential equation, we are done.

In [Ince, p. 496] a similar procedure is applied (for second order equations): Assume that $a \in \mathbb{C}$ is a regular singularity with exponents ρ and $\rho + \frac{1}{2}$. Multiply the variable y in the differential equation by $(x-a)^{-\rho}$, set $z = (x-a)^{-\rho}y$, and get a differential equation for z whose exponents are now 0 and $\frac{1}{2}$ [please check this!]. This can be done simultaneously for all singularities in $\mathbb{C}$, regardless of the type of the singularity at ∞, whose exponents do not change under the transformation. Of course, from a solution $z(x)$ of the new differential equation the solution of the original equation can easily be reconstructed via $y(x) = (x-a)^{\rho}z(x)$.

Landau indicates instances already observed by Schwarz for the algebraicity of all the solutions expressed in terms of the parameters a, b, c and the differences $c-a$, $c-b$. He uses Eisenstein's theorem to deduce these conditions in a mostly computational manner, see also [Höpp].

(B10' 2') The general hypergeometric equation. Write an n-th order differential operator L as an operator in $\delta = x\partial_x$,

$$L_\delta = \delta^n + q_1(x)\delta^{n-1} + \ldots + q_{n-1}(x)\delta + q_n(x),$$

with $q_i \in \mathbb{C}(x)$ rational functions. It has regular singularities in 0, 1 and ∞ and is non-singular elsewhere if and only if [Beukers-Heckman, Prop. 2.1, p. 327]

$$q_i(x) = \textstyle\sum_{j=0}^{i} c_{ij}(x-1)^{-j}, \qquad \text{for } c_{ij} \in \mathbb{C}.$$

It is called *hypergeometric* if

$$q_i(x) = c_{i0} + c_{i1}(x-1)^{-1}$$

for all i, i.e., if the poles of q_i at $x = 1$ have at most order 1. In this case, one may factor $(1-x)L$ into [Beukers-Heckman, p. 327]

$$(1-x)L = (\delta + \beta_1 - 1)\cdots(\delta + \beta_n - 1) - x(\delta + \alpha_1)\cdots(\delta + \alpha_n)$$

with $\alpha_i, \beta_i \in \mathbb{C}$. The local exponents at 0, 1 and ∞ are $1-\beta_1, ..., 1-\beta_n$ at $x = 0$, $\alpha_1, ..., \alpha_n$ at $x = \infty$, and $0, 1, ..., n-2$ and $\sum_{i=1}^n \beta_i - \sum_{i=1}^n \alpha_i$ at $x = 1$. If $\beta_1, ..., \beta_n$ are pairwise not congruent modulo $\mathbb{Z}$, a basis of solutions of $Ly = 0$ is given by

$$y_i(x) = x^{1-\beta_i}{}_nF_{n-1}(1+\alpha_1-\beta_i, ..., 1+\alpha_n-\beta_i; 1+\beta_1-\beta_i, ..., \widehat{1}, ..., 1+\beta_n-\beta_i; x),$$

where $1 + \beta_i - \beta_i = 1$ is omitted and where

$$
{}_nF_{n-1}(a_1,...,a_n;b_1,...,b_{n-1};x)=\sum_{k=0}^{\infty}\frac{a_1^{\bar{k}}\cdots a_n^{\bar{k}}}{b_1^{\bar{k}}\cdots b_{n-1}^{\bar{k}}n!}x^k.
$$

(B11 3) Here is an example of a (regular) singular differential equation whose solutions are nevertheless nice: Take $x^2y''-3xy'+3y=0$. This is an Euler equation. The indicial polynomial is $\rho(\rho-1)-3\rho+3=\rho^2-4\rho+3=(\rho-1)(\rho-3)$. The solutions are spanned by $y_1(x)=x$, $y_2(x)=x^3$, hence holomorphic despite the presence of the singularity. The singularities of differential equations which admit a basis of locally holomorphic solutions are called $apparent\ singularities$. They are kind of "harmless".

(B12 4) In general, the local solutions of a regular singular differential equation are no longer power series (not even formal ones). Take $x^2y''-xy'+y=0$ with indicial polynomial $(\rho-1)^2$ and solutions x and $x\log(x)$. And the equation $xy'-\alpha y=0$ has local solution cx^α, $c\in\mathbb{C}$, for any $\alpha\in\mathbb{C}$. For $\alpha\notin\mathbb{Z}$, this defines a "multivalued" function $x^\alpha=\exp(\alpha\log(x))$ at 0.

(B13 5) The Legendre differential equation

$$
4t(t-1)z''+4(2t-1)z'+z=0
$$

is associated to the family of elliptic curves $E_t: y^2=x(x-1)(x-t)$, $t\in\mathbb{C}$, by integrating the (essentially) unique holomorphic 1-form

$$
\omega_t=\frac{dx}{y}=\frac{dy}{(x(x-1)(x-t))^{1/2}}
$$

on E_t. Then the integral $z(t)=\int_\gamma\omega_t$ of ω_t along a closed path γ on E_t satisfies the differential equation [...the path has to be varied continuously with t, but this does not affect the integral]. The equation has clearly regular singularities at 0 and 1, but what about ∞?

(B14 6) The Bessel equation is $x^2y''+xy'+(x^2-\alpha^2)y=0$ with $\alpha\in\mathbb{C}$. For $\alpha\neq 0$, it has a regular singularity at 0. At ∞, the transformed equation is $x^4y''+x^3y'+(1-\alpha^2x^2)y=0$, hence ∞ is an irregular singularity of the Bessel equation.

Exponents at 0 are $\pm\alpha$, first local solution $y_1(x)=x^\alpha\sum_{i=0}^\infty c_ix^i$, $c_0=1$, with linear recursion $i(i+2\alpha)c_i+c_{i-2}=0$, $c_i=0$ for i odd. This is the Bessel function. Second solution (for $\alpha\neq 0$) is more complicated and involves harmonic numbers $h_j=\sum_{k=1}^i\frac{1}{k}$ and the Euler-Mascheroni constant $\gamma=\lim_{i\to\infty}(h_i-\log(i))=0.577216...$; it is of the form $y_2(x)=x^{-n}z(x)+c\log(x)y_1(x)$. The case $\alpha=0$ has to be treated separately.

The Bessel functions arise naturally when solving the Poisson equation for a system with cylindrical symmetry.

[physics stackexchange: https://physics.stackexchange.com/questions/145177/what-physical-phenomena-are-modelled-by-chebyshev-equation]

(B15 7) Apéry's differential equation is of the form

$$
(x^4-34x^3+x^2)y'''+(6x^3-153x^2+3x)y''+(7x^2-112x+1)y'+(x-5)y=0.
$$

It has four regular singularities, at 0, ∞ and $(1\pm\sqrt{2})^4$ (see example (P7)). The associated linear recursion has order 2, with cubic coefficients,

$$
k^3c_k=(34k^3-51k^2+27k-5)c_{k-1}-(k-1)^3c_{k-2}.
$$

For initial values $c_0 = 1$ and $c_1 = 5$ one obtains integer values $c_k = \sum_{i=0}^k \binom{k+i}{i}^2 \binom{k}{i}^2$. For initial values $c_0 = 0$ and $c_1 = 6$ one obtains only that $\mathrm{lcm}(1, 2, ..., k)^3 c_k$ is integral, while c_k itself is not globally bounded (lcm denotes the lowest common multiple).

As a matter of curiosity, the square-root $\sqrt{y(x)}$ of a solution to Apéry's equation satisfies a differential equation of second order, namely

$$(x^3 - 34x^2 + x)y'' + (2x^2 - 51x + 1)y' + \tfrac{1}{4}(x - 10)y = 0.$$

One says that Apéry's equation is the *square* of the latter equation. The second order equation has the same four regular singularities, at 0, ∞, and $(1 \pm \sqrt{2})^4$. The respective linear recursion is

$$k^2 c_k = (34k^2 - 51k + 39/2)c_{k-1} - (k - 3/2)^2 c_{k-2}.$$

(B16 8) The Airy equation $y'' - xy = 0$ (George Biddell Airy, 1801-1892, article on optics from 1838) has a unique singular point, namely at ∞. The local form at ∞ corresponds to the equation $x^5 y'' + 2x^4 y' - y = 0$ at 0. Setting $Y = (y, y')^T$, we get the equivalent system of first order linear differential equations

$$Y' = \begin{pmatrix} 0 & 1 \\ x^{-5} & -2x^{-1} \end{pmatrix} \cdot Y$$

representing the Airy equation at ∞. A fundamental matrix of solutions of this system (now considered at 0) is

$$Y(x) = \Phi(x)x^J U e^{Q(\sqrt{x})}$$

with

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, J = \begin{pmatrix} 1/4 & 0 \\ 0 & -3/4 \end{pmatrix}, Q = \begin{pmatrix} -2/3x^{3/2} & 0 \\ 0 & 2/3x^{3/2} \end{pmatrix},$$

and some function $\Phi(x)$. Outside ∞, the local solutions are surprisingly complicated: [Mahaffy] gives as solutions at 0 the expansions

$$\begin{aligned} y(x) &= c_0 \sum_{k=0}^{\infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3k-1)3k} x^{3k} + c_1 \sum_{k=0}^{\infty} \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots 3k(3k+1)} x^{3k+1} \\ &= c_0 Ai(x) + c_1 Bi(x) \end{aligned}$$

with the Airy functions

$$Ai(x) = \frac{1}{\pi} \int_0^\infty \cos(\frac{1}{3}z^3 + xz)dz,$$

$$Bi(x) = \frac{1}{\pi} \int_0^\infty \exp(-\frac{1}{3}z^3 + xz) + \sin(\frac{1}{3}z^3 + xz)dz,$$

For a graphical presentation, see [Mahaffy, p. 8]. Both functions oscillate on the negative real axis, while on the positive real axis Ai tends to 0 and Bi to ∞. Airy's equation is related to the one-dimensional time independent Schrödinger equation with total energy E

$$-\frac{\hbar}{2m} y'' + V(x)y = Ey.$$

This equation becomes for the special potential $V(x) = x$ the (modified) Airy equation

$$y'' - \frac{2m}{\hbar}(x - E)y = 0.$$

(B17 9) The operators $L_1 = x^2\partial^2 - x\partial - x^3$, $L_2 = x^2\partial^2 - x\partial - x^2$ and $L_3 = x^2\partial^2 - x\partial - x$ have the same initial form $\mathrm{in}(L_i) = x^2\partial^2 - x\partial$ but show quite different behaviour when one tries to find normal forms for them and to compute their power series solutions [Gann-Hauser, ex. 1, 1^{bis}, 1^{ter}, p. 14].

(B18 10) Here are four second order differential equations with four regular singular points admitting at least one power series solution with integral coefficients [ChCh2, p. 20],

$$x(x^2-1)y'' + (3x^2-1)y' + xy = 0.$$

$$x(x^2+3x+3)y'' + (3x^2+6x+3)y' + (x+1)y = 0.$$

$$x(x-1)(x+8)y'' + (3x^2-14x-8)y' + (x+2)y = 0.$$

$$x(x^2+11x-1)y'' + (3x^2+22x-1)y' + (x+3)y = 0.$$

In general, it seems to be extremely difficult to detect from the differential equation whether there exists a solution with integer coefficients (for a suitable choice of initial values). Apéry's equation is such an example. Zagier made a whole search for further examples. Among a 100 million computed cases of Apéry type equations, he found only seventeen equations with integral solutions.

(B19 11) $(x^2-b^2)(x^2-c^2)y'' + x(x^2-b^2+x^2-c^2)y' - [m(m+1)x^2 - (b^2+c^2)p]y = 0$ Lamé's equation.

(B20 12) Legendre's equation, with eigenvalue λ. Solutions can be extended into singularity if and only if $\lambda = n(n+1)$, and the solutions are then the associated Legendre polynomials. The equation arises naturally when solving the Poisson equation for a system with spherical symmetry (such as the hydrogen atom). Legendre's equation occurs quite often in areas such as electrodynamics and quantum mechanics.

[physics stackexchange: https://physics.stackexchange.com/questions/145177/what-physical-phenomena-are-modelled-by-chebyshev-equation]

(B21 13) $(x^2-1)y'' + xy' - \lambda^2 y = 0$ Chebyshev's equation. Regular singularities at ± 1 and ∞. Recursion $c_{i+2} = \frac{i^2-\lambda^2}{(i+2)(i+1)}c_i$. Solutions involve Chebyshev polynomials of first and second kind.

[physics stackexchange: https://physics.stackexchange.com/questions/145177/what-physical-phenomena-are-modelled-by-chebyshev-equation]

(B22 14) $y'' - 2xy' + \lambda y = 0$,

Hermite's equation. Irregular singularity at ∞. Solutions at 0 are holomorphic, can be expressed as linear combinations of two hypergeometric series, the second being the Hermite polynomial H_n if $\lambda = 2n \in 2\mathbb{N}$. The recursion for the coefficients is $c_{i+2} = \frac{2i-\lambda}{(i+2)(i+1)}c_i$.

[https://mathworld.wolfram.com/HermiteDifferentialEquation.html]

$H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$, ...

Exponential generating function: $\sum_{k=1}^{\infty} H_k(x)\frac{1}{k!}t^k = e^{2xt-t^2}$.

(B23 15) $xy'' + (\nu + 1 - x)y' + \lambda y = 0$,

Laguerre's equation. Regular singularity at 0 and irregular singularity at ∞. If $\lambda \in \mathbb{N}$, the solution at 0 is polynomial and thus extends into 0, giving the associated Laguerre polynomial for arbitary n, and the (classical) Laguerre polynomial for $\nu = 0$. The recursion for the coefficients is $c_{i+1} = \frac{i-\lambda}{(i+1)(i+\nu+1)}c_i$.

[Wolfram: https://archive.lib.msu.edu/crcmath/math/math/l/l039.htm]

Physics Forum: The equation arises in solving Schrödinger's equation to find the quantum-mechanical wave function of hydrogen. Specifically, it's associated with the radial part of the wave function.

[https://www.physicsforums.com/threads/uses-of-laguerre-differential-equ.222949/]

(B24 16) $xy'' + (c - x)y' - ay = 0$,

Kummer's equation (confluent hypergeometric equation). Has regular singularity at 0 and irregular singularity at ∞. It is called *confluent* since in the hypergeometric equation with three singular points two are merged to one singularity.

[https://math.stackexchange.com/questions/190486/transforming-differential-equation-to-a-kummers-equation]

(B25 17) $(1 - x^2)y'' - 2(\mu + 1)xy' + (\nu - \mu)(\nu + \mu + 1)y = 0$,

Gegenbauer's equation, singularities at ± 1. If $-1/2 + \mu + \nu$ is an integer n, one of the solutions is the Gegenbauer polynomial $C_n(x)$.