FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART II: BASICS

Singularities of differential equations. A point $a \in \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called a *singularity* of a monic complex linear differential equation with meromorphic coefficients $Ly = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$ if at least one of the coefficients $p_i(x)$ has a pole at a. If all p_i are holomorphic at a, then a is a *non-singular* point of L. A singularity is called *regular* if it has a basis of local solutions which have moderate growth as one approaches the singular point. This is for the moment just an intuitive definition which still has to be made precise (also, the concept of basis of solutions is not specified rigorously yet). We will show later that regularity is equivalent to saying that for every $i = 1, ..., n$, the order of the pole of $p_i(x)$ at a is at most i. A singular point which is not regular is called an *irregular singularity.*

A point $a \in \mathbb{C} \cup \{\infty\}$ is called *apparent singularity* of L if $Ly = 0$ possesses locally at a a basis of holomorphic solutions. In example 4 from above, 0 is an apparent singularity, since the local solutions at 0 are polynomials.

Let $L = \sum_{j=0}^n p_j \partial^j = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij} x^i \partial^j$ be a differential operator with polynomial, holomorphic or formal coefficients $p_j(x) = \sum_{i=0}^\infty c_{ij} x^i$. The *initial form* (or initial operator) of L at 0 is the operator

$$L_0 = \mathrm{in}(L) = \textstyle\sum_{i-j=\tau} c_{ij} x^i \partial^j,$$

where $\tau = \min_{c_{ij} \neq 0} \{i - j\} \in \mathbb{Z}$ is the *shift* of L (or rather of L_0) at 0. The polynomial

$$\chi_L(r) = \textstyle\sum_{i-j=\tau} c_{ij} r^{\underline{j}},$$

where $r^{\underline{j}} = r(r-1)\cdots(r-j+1)$ denotes the *falling factorial*, is called the *indicial* or *characteristic polynomial* of L at 0. Cleary, $\chi_L = \chi_{L_0}$. Analogous definitions hold at other points $a \in \mathbb{P}^1_{\mathbb{C}}$, taking into account the respective Taylor expansions of the coefficients of L at a, replacing x by $x - a$ if $a \in \mathbb{C}$ and leaving ∂ invariant. For $a = \infty$, one has to replace in the coefficients of L the variable x by $\frac{1}{x}$ and adjust accordingly the derivations ∂^i, using the differentiation rules for $\partial^i[f(\frac{1}{x})]$. In particular, ∂ has to be replaced by $-\frac{1}{x^2}\partial$. The resulting operator has then to be considered at 0 (see below).

If a is a non-singular point or a regular singularity of the differential operator L, then the complex roots of the indicial polynomial χ_L of L at a are called the *local exponents* of L at a.

Fact. *The point* 0 *is a non-singular point or a regular singularity of* L *if and only if the initial form* L_0 *of* L *has the same order as* L.

Proof. The assertion is easily checked using the order condition on the poles from above. An extensive characterization of regular singularities will be provided later on. ⥀

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Expansion at infinity. If $L = \sum_{j=0}^{n}\sum_{i=0}^{\infty} c_{ij}x^i\partial^j$ is a differential operator on $\mathbb{P}^1_{\mathbb{C}}$ with meromorphic coefficients p_i we may expand L in a local chart at ∞. To this end, replace x by $\frac{1}{x}$ in the coefficients $p_i(x)$ and in the solutions $y(x)$ of $Ly = 0$. Let us call ψ this automorphism of $\mathbb{P}^1_{\mathbb{C}}$. Applying ψ results in a change of the differential operator L to an operator $\psi^*(L)$ – the *pullback* of L under ψ – where now also the derivations ∂^i will have to be adapted. In fact, for a function f, the i-th derivative $\partial^i[f(\frac{1}{x})] = \partial^i[f(\psi(x))]$ can be expressed as the composition of a differential operator $L_i = \psi^*(\partial^i)$ applied to $f(x)$ with the subsequent substitution of x by $\frac{1}{x}$, say, $\partial^i[f(\frac{1}{x})] = (L_i f)(\frac{1}{x})$. In particular, we will have

$$L_1 = -\frac{1}{x^2}\partial, \qquad \text{since } \partial[f(\tfrac{1}{x})] = -\tfrac{1}{x^2}(\partial f)(\tfrac{1}{x}),$$

$$L_2 = \frac{1}{x^4}\partial^2 + 2\frac{1}{x^3}\partial, \qquad \text{since } \partial^2[f(\tfrac{1}{x})] = \tfrac{1}{x^4}(\partial^2 f)(\tfrac{1}{x}) + 2\tfrac{1}{x^3}(\partial f)(\tfrac{1}{x}),$$

$$L_3 = -\frac{1}{x^6}\partial^2 - 6\frac{1}{x^5}\partial^2 - 6\frac{1}{x^4}\partial,$$

$$\text{since } \partial^3[f(\tfrac{1}{x})] = -\tfrac{1}{x^6}(\partial^2 f)(\tfrac{1}{x}) - 4\tfrac{1}{x^5}(\partial^2 f)(\tfrac{1}{x}) - 2\tfrac{1}{x^5}(\partial^2 f)(\tfrac{1}{x}) - 6\tfrac{1}{x^4}(\partial f)(\tfrac{1}{x}).$$

Example. (I11) Let us take the operator $L = x^3\partial^2 - (x - x^2)\partial + 1$. It has initial form $L_0 = -x\partial + 1$ at the origin 0 of $\mathbb{C}$. As its order is smaller than the order of L, the point 0 is an irregular singularity of L. For instance, $y(x) = \sum_{k=0}^{\infty} k!x^{k+1}$ is a divergent formal power series solution of $Ly = 0$. Let us compute the expansion of L at ∞. Substitution gives

$$\psi^*(L) = \frac{1}{x^3}\frac{1}{x^4}\partial^2 + 2\frac{1}{x^3}\frac{1}{x^3}\partial - (\frac{1}{x} - \frac{1}{x^2})(-\frac{1}{x^2})\partial + 1 =$$
$$\frac{1}{x^7}\partial^2 + [2\frac{1}{x^6}\partial + \frac{1}{x^3} - \frac{1}{x^4}]\partial + 1.$$

Multiplication with the common denominator x^7 results in the operator

$$\widetilde{L} = \partial^2 + [2x + x^4 - x^3]\partial + x^7,$$

which is non-singular at 0. This shows that the local structure of a differential equation at ∞ may not be immediately obvious from the expansion of the operator at 0.

Systems of linear differential equations. Let (K, ∂) be a differential field, i.e., a field together with a derivation $\partial : K \to K$ (think of K the field of meromorphic functions and $\partial = \frac{d}{dx}$ the usual derivative). The field of constants $C \subset K$ consists of the elements f with $\partial f = 0$. A system of n linear first order equations over K is of the form

$$\begin{aligned} y_1' &= a_{11}y_1 + \cdots + a_{1n}y_n \\ &\vdots \qquad \vdots \qquad\qquad \vdots \\ y_n' &= a_{n1}y_1 + \cdots + a_{nn}y_n \end{aligned}$$

or, in matrix notation, $\partial Y = AY$, with the unknown column vector $Y = (Y_1, Y_2, ..., Y_n)^T$ and an $(n \times n)$ matrix $A = (a_{ij}) \in \mathrm{M}_n(K)$. Here ∂ acts on K^n componentswise, i.e., $\partial(Y_1, ..., Y_n)^T = (\partial Y_1, ..., \partial Y_n)^T$. The induced linear map is

$$L = \partial - A : K^n \to K^n,$$
$$Y \to \partial Y - AY.$$

If $K \subset K'$ is a field extension with $\partial' : K' \to K'$ extending $\partial : K \to K$, any vector $Y \in K'^n$ such that $\partial Y = AY$ is called a solution of $\partial Y = AY$ in K'. An $(n \times n)$ invertible matrix $\Phi \in \mathrm{GL}_n(K')$ with $\Phi' = A\Phi$ is called a *fundamental solution matrix* of $Y' = AY$ in K'.

Remark. We will see later that the set $\mathrm{Sol}_K := \{Y \in K^n, \partial Y = AY\}$ is a vector space of dimension $\le n$ over the field of constants C of K. In general, $\dim_C(\mathrm{Sol}_K) < n$. However, there always exists a differential field extension $K \subset K'$ such that over K' the solution space has dimension n. Such extensions are known, if they are minimal, as Picard-Vessiot extensions [vdPS].

Example. (I13) Let K be one of the fields $\mathbb{C}\{\{x\}\} = \mathrm{Quot}(\mathbb{C}\{x\})$ or $\mathbb{C}((x)) = \mathrm{Quot}(\mathbb{C}[[x]])$ equipped with the derivation $\partial : K \to K$ defined by $\partial x = 1$ and set $\delta = x\partial : K \to K$, $\delta x = x$. If $A \in \mathrm{M}_n(K)$, we obtain the maps

$$\begin{aligned} \partial - A : K^n &\to K^n, \\ Y &\mapsto Y' - AY, \end{aligned}$$

and

$$\begin{aligned} \delta - A : K^n &\to K^n, \\ Y &\mapsto xY' - AY. \end{aligned}$$

For example, consider the system $Y' = AY$ with $A = \begin{pmatrix} 0 & 1 \\ \frac{-2}{x} & \frac{x+2}{x} \end{pmatrix}$. We have $Y' = AY$ if and only if $y_1' = y_2$ and $y_2' = -\frac{2}{x}y_1 + \frac{(x+2)}{x}y_2$. This gives, setting $y = y_1$ and $y' = y_2$ [exercise: check the formulas] the scalar equation $y'' = -\frac{2}{x}y + \frac{(x+2)}{x}y'$, say $xy'' - (x+2)y' + 2y = 0$. Then $y_1(x) = e^x$ and $y_2(x) = 1 + x + \frac{1}{2}x^2$ are $\mathbb{C}$-linearly independent solutions of this equation. Therefore

$$\begin{pmatrix} e^x \\ e^x \end{pmatrix}$$

and

$$\begin{pmatrix} 1 + x + \frac{1}{2}x^2 \\ 1 + x \end{pmatrix}$$

are linearly independent solutions of $Y' = AY$ and hence $\Phi = \begin{pmatrix} e^x & 1 + x + \frac{1}{2}x^2 \\ e^x & 1 + x \end{pmatrix}$ is a fundamental solution matrix of the system $Y' = AY$.

The *singularities* of the system $\partial Y = AY$ are the poles of the entries of A. Similarly as for scalar equations, a singularity of a system is called *apparent* if there exists a fundamental solution matrix $\widetilde{Y}(x)$ of $\partial Y = AY$ with holomorphic entries.

Remark. Note that if we replace Y by PY in the system, where $P \in \mathrm{GL}_n(K)$ is an invertible matrix, we obtain a new system

$$\partial Y = (P^{-1}AP - P^{-1}\partial P)Y =: BY,$$

with $B = P^{-1}AP - P^{-1}\partial P$. Two systems $\partial Y = AY$ and $\partial Y = BY$ are called *gauge equivalent* (over K) if there exists $P \in \mathrm{GL}_n(K)$ so that $B = P^{-1}AP - P^{-1}\partial P$. If $P \in \mathrm{GL}_n(K')$ for some differential field extension K' of K then $\partial Y = AY$ and $\partial Y = BY$ are called gauge equivalent over K'.

Expressed in terms of maps we get from $(P^{-1} \circ \partial \circ P)(Y) = P^{-1}(\partial(PY)) = P^{-1}(\partial P)Y + P^{-1}P\partial Y = P^{-1}(\partial P)Y + \partial Y = (P^{-1}\partial P + \partial)Y$ that

$$\begin{aligned} \partial - B = P^{-1} \circ (\partial - A) \circ P &= P^{-1} \circ \partial \circ P - P^{-1} \circ A \circ P \\ &= P^{-1}\partial P + \partial - P^{-1}AP \\ &= \partial - (P^{-1}AP - P^{-1}\partial P) \\ &= P^{-1}AP - P^{-1}\partial P. \end{aligned}$$

Lemma (Jósef Maria Hoëné-Wroński, 1776-1853) *Let be given n holomorphic functions $y_1, ..., y_n$ defined in a neighborhood of $0 \in \mathbb{C}$. They are $\mathbb{C}$-linearly dependent if and only if the* Wronskian matrix

$$W(y_1, ..., y_n) = \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \cdot & & \cdot \\ \cdot & & \cdot \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

formed by the row vector $(y_1, ..., y_n)$ and its first $n-1$ derivatives has zero determinant.

Remarks. (a) This can be done as an exercise, using induction on n, see below. See also [Honda, Lemma, p. 172, resp. Kol1] for a characteristic p version: If the determinant of Wronskian vanishes, then $y_1, ..., y_n$ are linearly dependent over $K(x^p)$. See also appendix B in [diVi2].

(b) The case $n = 2$ is particularly instructive. Let $y = y(x)$ and $z = z(x)$ be two holomorphic functions, not identically 0, and assume that their Wronskian determinant is 0,

$$\det \begin{pmatrix} y & z \\ y' & z' \end{pmatrix} = yz' - y'z = 0.$$

This is equivalent to $\frac{y}{y'} = \frac{z}{z'}$ and thus to $\log(y)' = \log(z)'$. We get equivalently $\log y = \log z + c$ for some constant $c \in \mathbb{C}$, hence $y = e^c \cdot z$ as claimed.

Proof. We prove the non-trivial implication. It is a bit tricky. So let $W(y_1, ..., y_n)$ have zero determinant. If $n = 1$, we get $y_1 = 0$. We now assume $n \geq 2$, and wlog that $y_n \neq 0$. One checks by computation that

$$\det(W(y_1, ..., y_n)) = y_n^n \cdot \det(W(y_1/y_n, ..., y_{n-1}/y_n, 1)),$$

and

$$\det(W(y_1/y_n, ..., y_{n-1}/y_n, 1)) = (-1)^n \cdot \det(W((y_1/y_n)', ..., (y_{n-1}/y_n)')).$$

Applying the lemma in the case $n-1$ we get constants $c_1, ..., c_{n-1} \in \mathbb{C}$ such that

$$c_1 \cdot (y_1/y_n)' + \cdots + c_{n-1} \cdot (y_{n-1}/y_n)' = 0.$$

It follows that

$$c_1 \cdot y_1/y_n + \cdots + c_{n-1} \cdot y_{n-1}/y_n = c,$$

for some $c \in \mathbb{C}$. This shows that $y_1, ..., y_n$ are $\mathbb{C}$-linearly dependent. ◯

Corollary. *An n-th order linear differential equation $Ly = 0$ with holomorphic coefficients has at most n $\mathbb{C}$-linearly independent local holomorphic solutions.*

Proof. Assume we had $n+1$ solutions $y_1, ..., y_{n+1}$. The columns of $W(y_1, ..., y_{n+1})$ are given by $(y_i, y_i', ..., y_i^{(n-1)}, y_i^{(n)})^T$. The entries of each of these vectors are $\mathbb{C}\{x\}$-linearly dependent since they satisfy the linear relation given by $Ly = 0$. It follows that the determinant of $W(y_1, ..., y_{n+1})$ is zero. By Wronski's lemma we conclude that $y_1, ..., y_{n+1}$ are $\mathbb{C}$-linearly dependent. ◯

Lemma. *Consider two n-th order linear differential equations $Ly = 0$ and $My = 0$. Assume given holomorphic functions $y_1, ..., y_n$ at 0 which form a basis of solutions for both L and M. Then there exists a meromorphic function h at 0 such that $M = h \cdot L$.*

Proof. It is sufficient to prove the assertion for formal power series operators. The convergent case goes along the same lines. Let $\mathbb{C}((x))$ denote the quotient field of $\mathbb{C}[[x]]$, i.e., the field of formal Laurent series.

Define a map $\alpha : \mathbb{C}((x))[\partial] \to \mathbb{C}((x))^n$, sending a differential operator N to the vector $(Ny_1, ..., Ny_n)$ given by evaluation. By definition, L and M belong to the kernel of α. But $\mathbb{C}((x))[\partial]$ is a polynomial ring over a field and hence a principal ideal domain. Hence $\mathrm{Ker}(\alpha)$ is generated by one operator N, and L and M are $\mathbb{C}((x))[\partial]$-multiples of it. But $y_1, ..., y_n$ are then also $\mathbb{C}$-linearly independent solutions of N, therefore N has order at least n. As L and M are multiples of it (as elements of the ring $\mathbb{C}((x))[\partial]$), N must have order n. This implies that $L = f \cdot N$, $M = g \cdot N$ for suitable $f, g \in \mathbb{C}((x))$. Setting $h = f/g$ we get $M = g \cdot L$ as required. ↺