FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART III: EULER EQUATIONS

The objective of this section is to describe the local solutions of Euler differential equations $L_0 y = 0$ at 0 in the case where a local exponent ρ is a *multiple* root of the indicial polynomial χ_{L_0}.

The situation for general linear differential equations $Ly = 0$ (always assuming the singularities to be regular) will be discussed in a later chapter. The construction of solutions of $Ly = 0$ goes back to Fuchs and Frobenius. The latter cites in [Frob1] two papers of Fuchs as predecessors of his investigation [Fuchs1, Fuchs2] as well as a paper of Thomé [Thom1], p. 200, shortening the proof of Fuchs, see also [Thom2, Thom3]. In [Mez, p. 58] the author attributes the first description of the solutions and the use of variation of constants to Fuchs, while Frobenius improved and simplified Fuchs' construction by treating the solutions involving logarithms directly. See [Gray, J.: Linear differential equations and group theory from Riemann to Poincaré. Birkhäuser, 2000] for a historical account.

Ince [Ince, footnote, p. 396] reproduces quite accurately their methods, see also section 4.3 in [Mez]. You may also consult [Teschl, section 4.4, p. 134] for an exposition of Frobenius' method. Mezzarobba presents also another method to construct solutions, developed apparently by Heffter in 1894 and exposed in the book of Poole from 1936, see [Mez, section 4.4, Poole, V.16]. We will present here a slightly modernized version of the Frobenius story. Some astonishing turns will enrich our journey.

It turns out that from now on logarithms will appear. As these are no longer holomorphic at 0 and thus do not admit a power series expansion, we have to enlarge our space of functions $\mathbb{C}\{x\}$, $\mathbb{C}[[x]]$ or $\mathbb{C}((x))$ so as to include also powers $\log(x)^k$. We do this universally by adjoining a variable z which will mimic the role of $\log(x)$. In the sequel, $\mathbb{C}((x))$ will denote a field of formal Laurent series with monomials whose exponents may even be complex numbers. In order to have a well defined multiplication, we restrict to the field generated by series of the form $h = x^\rho \sum_{i=0}^\infty c_i x^i$ with $\rho \in \mathbb{C}$ and $i \in \mathbb{N}$. We will neglect in this section convergence questions and only work formally. Monomials x^ρ with complex exponents $\rho \in \mathbb{C}$ will not do any harm: differentiation is defined as usual, $\partial x^\rho = \rho x^{\rho-1}$. Integration is given by $\int x^\rho = \frac{1}{\rho+1} x^{\rho+1}$ provided that $\rho \neq -1$.

We denote by $\mathbb{C}((x))[z]$ the ring of polynomials in a new variable z with coefficients in $\mathbb{C}((x))$. Working with $\mathbb{C}((x))[z]$ instead of the polynomial ring $\mathbb{C}((x))[\log(x)]$ in $\log(x)$ has notational advantages – the substance is of course the same. Compare with page 184 in [Honda, T.: Algebraic differential equations, Symp. Math. 24, 169-204. Academic Press 1981]. We consider $\mathbb{C}((x))[z]$ as a differential ring via the derivation

$$\underline{\partial} : \mathbb{C}((x))[z] \to \mathbb{C}((x))[z],$$

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$$\underline{\partial}(x^i) = \partial(x^i) = ix^{i-1}, \qquad \underline{\partial} z = x^{-1},$$

$$\underline{\partial}(x^i z^k) = (iz+k)x^{i-1}z^{k-1}.$$

This construction formalizes the differential ring $\mathbb{C}((x))[\log(x)]$ equipped with the usual differentiation operator ∂. We shall call $\underline{\partial}$ the *logarithmic extension* of ∂ to $\mathbb{C}((x))[z]$. We claim to have an isomorphism of differential rings

$$((\mathbb{C}((x))[z], \underline{\partial}) \to \mathbb{C}((x))[\log(x)], \partial),\ z \to \log(x).$$

Indeed, the map is linear, surjective and, as $\partial(\log(x)) = (\log(x))' = x^{-1}$, compatible with the derivations $\underline{\partial}$ and ∂. It is also injective since a relation $\sum_{k=0}^m h_k(x)\log(x)^k = 0$ with $h_k \in \mathbb{C}((x))$ implies that all h_k are 0. This last part is due to the fact known from analysis that $\log(x)$ is transcendental over $\mathbb{C}((x))$, i.e., does not satisfy any polynomial relation with coefficients in $\mathbb{C}((x))$. We have the differentiation rule

$$\underline{\partial}(\textstyle\sum_{k=0}^m h_k(x)z^k) = \sum_{k=0}^m \partial h_k(x)z^k + \sum_{k=1}^m kh_k(x)x^{-1}z^{k-1}.$$

Note that $\underline{\partial}$ does not increase the degree in z of polynomials in $\mathbb{C}((x))[z]$. If we denote by $\mathbb{C}((x))[z] = \bigoplus_{k=0}^\infty \mathbb{C}((x))[z]_k$ the natural grading defined by the degree in z, we get by restriction maps

$$\underline{\partial} : \mathbb{C}((x))[z]_k \to \mathbb{C}((x))[z]_k \oplus \mathbb{C}((x))[z]_{k-1}.$$

We may thus write

$$\underline{\partial} = \partial + \theta_z : \mathbb{C}((x))[z] \to \mathbb{C}((x))[z],$$

where $\partial(h(x)z^k)) = \partial(h(x))z^k = h'(x)z^k$ and where the map θ_u defined by

$$\theta_z(h(x)z^k) = h(x)\underline{\partial}(z^k) = kx^{-1}h(x)z^{k-1}$$

sends $\mathbb{C}((x))[z]_k$ to $\mathbb{C}((x))[z]_{k-1}$ for $k \geq 1$, and is 0 on $\mathbb{C}((x))$. This definition reflects, of course, the differentiation rule for the logarithm,

$$\partial(h(x)\log(x)^k) = h'(x)\log(x)^k + kx^{-1}h(x)\log(x)^{k-1}.$$

The image of $\underline{\partial}$ is the $\mathbb{C}$-subspace H of $\mathbb{C}((x))[z]$ of polynomials $\sum_{k=0}^m h_k(x)z^k$ satisfying the integrability condition

$$\partial(xh_{k-1}(x)) = kh_k(x)$$

for all $k \geq 0$. Integration on $\mathbb{C}((x))$ is now defined by $\int x^\rho = \frac{1}{\rho+1}x^{\rho+1}$ for $\rho \neq -1$ and $\int x^{-1} = u$ [we set the additive constants equal to 0]. It is a map $\int : \mathbb{C}((x)) \to \mathbb{C}((x))[z]$ which can be trivially extended to the subspace H of $\mathbb{C}((x))[z]$. Thus $\int : H \to \mathbb{C}((x))[z]$ is a right inverse to $\underline{\partial}$, say $\underline{\partial} \circ \int = \mathrm{Id}_H$. Arbitrary elements $\sum_{k=0}^m h_k(x)z^k \in \mathbb{C}((x))[z]$ cannot be integrated in general, so $\int$ does not extend to a map $\mathbb{C}((x))[z] \to \mathbb{C}((x))[z]$ inverse to $\underline{\partial}$.

Let L be an n-th order linear differential operator on $\mathbb{C}((x))$,

$$L = p_0\partial^n + p_1\partial^{n-1} + \ldots + p_{n-1}\partial + p_n,$$

with coefficients p_i in $\mathbb{C}((x))$. We extend L to the operator $\underline{L}$ on $\mathbb{C}((x))[z]$ defined by

$$\underline{L} = p_0\underline{\partial}^n + p_1\underline{\partial}^{n-1} + \ldots + p_{n-1}\underline{\partial} + p_n.$$

$$\underline{L}(h(x)z^k) = \partial h(x)z^k + kh(x)x^{-1}z^{k-1}.$$

This operator is now compatible with the substitution of z by $\log(x)$:

Proposition. *Let L be a differential operator on $\mathbb{C}((x))$ with extension $\underline{L}$ to $\mathbb{C}((x))[z]$ as defined above. Let $\rho \in \mathbb{C}$, $k \in \mathbb{N}$, and $h(x) \in \mathbb{C}[[x]]$ a formal power series. Then*

$$L(x^\rho h(x)\log(x)^k) = \underline{L}(x^\rho h(x) z^k)_{|z=\log(x)}.$$

Proof. This holds by definition of $\underline{\partial}$ and since $\log(x)' = x^{-1}$. ⥀

Corollary. *The solutions of a differential equation $Ly = 0$ in $\mathbb{C}((x))[\log(x)]$ are in bijection with the solutions of the associated equation $\underline{L}y = 0$ in $\mathbb{C}((x))[z]$.* ⥀

Example. The Euler equation $x^2y'' + 3xy' + 1 = 0$ with operator $L = x^2\partial^2 + 3x\partial + 1$ has indicial polynomial $\chi_L = \rho(\rho-1) + 3\rho + 1 = (\rho+1)^2$ with double root $\rho = -1$. It is immediately checked that $y_1 = x^{-1}$ and $y_2 = x^{-1}\log(x)$ are solutions of $Ly = 0$. The operator $\underline{L} = x^2\underline{\partial}^2 + 3x\underline{\partial} + 1$ therefore has, as it should be, solutions x^{-1} and $x^{-1}u$. Indeed, $\underline{\partial}(x^{-1}z) = x^{-2}(-z+1)$ and

$$\underline{\partial}^2(x^{-1}z) = \underline{\partial}(x^{-2}(-z+1)) = -2x^{-3}(-z+1) - x^{-3} = x^{-3}(2z-3).$$

Thus,

$$\underline{L}(x^{-1}z) = x^{-1}(2z-3) + 3x^{-1}(-z+1) + x^{-1}z = x^{-1}(2z-3-3z+3+z) = 0.$$

The proposition and its corollary guarantee that when we search for logarithmic solutions of a differential equation $Ly = 0$ we may study instead the differential equation $\underline{L}y = 0$ on $\mathbb{C}((x))[z]$, with $\underline{L}$ associated to L as above. This notational trick simplifies substantially the formulation of the problem.

The evaluation of the induced linear map $\underline{L} : \mathbb{C}((x))[z] \to \mathbb{C}((x))[z]$ on elements $\sum h_k z^k$ requires a multiple application of the product rule, since each $\underline{\partial}^j = (\partial + \theta_z)^j$ is a j-fold composition. We will see that there evolves a precise pattern which we will explore next. We first concentrate on Euler operators.

Examples. (1) Let $L = \partial$ and $\underline{L} = \underline{\partial}$. Then

$$\underline{\partial}(x^i z^k) = ix^{i-1}z^k + x^i x^{-1} k z^{k-1} = (iz+k)x^{i-1}z^{k-1}.$$

(2) Let $L = \partial^2$, $\underline{L} = \underline{\partial}^2$. Then

$$\begin{aligned}
\underline{\partial}^2(x^i z^k) &= \underline{\partial}(ix^{i-1}z^k + kx^i x^{-1}z^{k-1}) \\
&= \underline{\partial}(ix^{i-1}z^k + kx^{i-1}z^{k-1}) \\
&= i^{\underline{2}}x^{i-2}z^k + kix^{i-2}z^{k-1} + k(i-1)x^{i-2}z^{k-1} + k^{\underline{2}}x^{i-2}z^{k-2} \\
&= i^{\underline{2}}x^{i-2}z^k + (2i-1)kx^{i-2}z^{k-1} + k^{\underline{2}}x^{i-2}z^{k-2} \\
&= i^{\underline{2}}x^{i-2}z^k + (i^{\underline{2}})'kx^{i-2}z^{k-1} + \tfrac{1}{2}(i^{\underline{2}})''k^{\underline{2}}x^{i-2}z^{k-2},
\end{aligned}$$

where $(t^{\underline{2}})' = (t(t-1))' = 2t-1$ and $(t^{\underline{2}})'' = (t(t-1))'' = 2$ denote the first and second derivatives of $t^{\underline{2}}$ with respect to the variable t. This computation suggests a general formula for $\underline{\partial}^j$. Here it is.

Lemma 1. *For $j, \ell \in \mathbb{N}$ and $\rho \in \mathbb{C}$, denote by $(\rho^{\underline{j}})^{(\ell)}$ the evaluation at $t = \rho$ of the ℓ-th derivative $(t^{\underline{j}})^{(\ell)} := \partial_t^\ell(t^{\underline{j}})$ of the falling factorial $t^{\underline{j}} = t(t-1)\cdots(t-j+1)$. Then*

$$\begin{aligned}
\underline{\partial}^j(x^\rho z^k) &= \rho^{\underline{j}}x^{\rho-j}z^k + (\rho^{\underline{j}})'kx^{\rho-j}z^{k-1} + \ldots + \tfrac{1}{j!}(\rho^{\underline{j}})^{(j)}k^{\underline{j}}x^{\rho-j}z^{k-j}. \\
&= [\rho^{\underline{j}}z^j + (\rho^{\underline{j}})'kz^{j-1} + \ldots + \tfrac{1}{j!}(\rho^{\underline{j}})^{(j)}k^{\underline{j}}] \cdot x^{\rho-j}z^{k-j}.
\end{aligned}$$

Remark. A similar formula appears in [Mezzarobba, Prop. 4.14 and 4.16, p. 69, 70].

Proof. To prove the formula, use induction on j and the following identities. ⥀

Lemma 2. *The derivatives of the falling factorials satisfy, for $j \in \mathbb{N}$, the identities*

$$
\begin{aligned}
&t^{\underline{j}} + (t^{\underline{j}})'(t-j) = (t^{\underline{j+1}})',\\
&(t^{\underline{j}})' + \tfrac{1}{2}(t^{\underline{j}})''(t-j) = \tfrac{1}{2}(t^{\underline{j+1}})'',\\
&\tfrac{1}{2}(t^{\underline{j}})'' + \tfrac{1}{6}(t^{\underline{j}})'''(t-j) = \tfrac{1}{6}(t^{\underline{j+1}})''',\\
&\ldots\\
&\tfrac{1}{(j-1)!}(t^{\underline{j}})^{(j-1)} + \tfrac{1}{j!}((t^{\underline{j}})^{(j)}(t-j) = \tfrac{1}{j!}(t^{\underline{j+1}})^{(j)},\\
&\tfrac{1}{j!}((t^{\underline{j}})^{(j)} = \tfrac{1}{(j+1)!}((t^{\underline{j+1}})^{(j+1)}.
\end{aligned}
$$

The general formula for $\ell = 0, ..., j-1$ is

$$\tfrac{1}{\ell!}(t^{\underline{j}})^{(\ell)} + \tfrac{1}{(\ell+1)!}((t^{\underline{j}})^{(\ell+1)}(t-j) = \tfrac{1}{(\ell+1)!}(t^{\underline{j+1}})^{(\ell+1)}.$$

Proof. The first equation follows directly from the product rule, say

$$(t^{\underline{j+1}})' = (t^{\underline{j}}(t-j))' = (t^{\underline{j}})'(t-j) + t^{\underline{j}}.$$

The other identities are proven by successive differentiation of the first equation. For instance, deriving the first equation gives

$$(t^{\underline{j}})' + (t^{\underline{j}})''(t-j) + (t^{\underline{j}})' = (t^{\underline{j+1}})'',$$

which is just the second equation. Differentiation of the equation for the general formula gives

$$\tfrac{1}{\ell!}(t^{\underline{j}})^{(\ell+1)} + \tfrac{1}{(\ell+1)!}((t^{\underline{j}})^{(\ell+2)}(t-j) + \tfrac{1}{(\ell+1)!}((t^{\underline{j}})^{(\ell+1)} = \tfrac{1}{(\ell+1)!}(t^{\underline{j+1}})^{(\ell+2)}.$$

Then use $\frac{1}{\ell!} + \frac{1}{(\ell+1)!} = \frac{\ell+2}{(\ell+1)!}$ to get the next equation

$$\tfrac{1}{(\ell+1)!}(t^{\underline{j}})^{(\ell+1)} + \tfrac{1}{(\ell+2)!}((t^{\underline{j}})^{(\ell+2)}(t-j) = \tfrac{1}{(\ell+2)!}(t^{\underline{j+1}})^{(\ell+2)}.$$

This proves the claim. ⟲

The next result, which follows from the above, will be the clue to understand why logarithms appear in the solutions of differential equations when the local exponents are multiple roots of the indicial polynomial.

Lemma 3. *Let $L_0 = \sum_{i=0}^n c_i x^i \partial^i$ be an Euler operator of shift 0, and let $\underline{L}_0 = \sum_{i=0}^n c_i x^i \underline{\partial}^i$ be the associated operator on $\mathbb{C}((x))[z]$. Denote by $\chi_0(\rho) = \sum_{i=0}^n c_i \rho^{\underline{i}}$ the indicial polynomial of L_0, and by $\chi_0^{(j)}$ its j-th derivative. Let $\rho \in \mathbb{C}$ and $k \in \mathbb{N}$. Then*

$$\underline{L}_0(x^\rho z^k) = x^\rho \cdot [\chi_0(\rho) z^k + \chi_0'(\rho) k z^{k-1} + \ldots + \tfrac{1}{n!}\chi_0^{(n)}(\rho) k^{\underline{n}} z^{k-n}].$$

Proof. This is a consequence of the formula for $\underline{\partial}^j(x^\rho z^k)$ in Lemma 1. ⟲

Remark. Note that for $k < n$ (which will be the relevant case) no negative powers of z appear in the expansion of $\underline{L}(x^\rho z^k)$ because $k^{\underline{j}} = 0$ for $j > k$. In this case the formula reduces to

$$\underline{L}(x^\rho z^k) = x^\rho \cdot [\chi_L(\rho) z^k + \chi_L'(\rho) k z^{k-1} + \ldots + \tfrac{1}{(k-1)!}\chi_L^{(k-1)}(\rho) k^{\underline{k-1}} z + \tfrac{1}{k!}\chi_L^{(k)}(\rho) k!].$$

Proposition. *If $\rho \in \mathbb{C}$ is an m-fold root of the indicial polynomial P_0 of an Euler operator L_0, then x^ρ, $x^\rho \log(x)$,..., $x^\rho \log(x)^{m-1}$ are solutions of $L_0 y = 0$.*

Proof. Indeed, ρ being an m-fold root of χ_0 signifies that $\chi_0^{(k)}(\rho) = 0$ for $k = 0, ..., m-1$, whence $\underline{L}_0(x^\rho z^k) = 0$. Substituting z by $\log(x)$ in this equation gives $L_0(x^\rho \log(x)^k) = 0$. ⟲

Corollary. *Let* $L_0 = \sum_{i-j=\tau} c_{ij} x^i \partial^j$ *be an Euler operator with indicial polynomial* $\chi_0(\rho) = \sum_{j=0}^{n} c_{ij} \underline{\rho^j}$. *Let* $\rho_1, ..., \rho_q \in \mathbb{C}$ *be the distinct roots of* χ_0, *each with multiplicity* $m_1, ..., m_q$, *respectively. Then* x^{ρ_i}, $x^{\rho_i} \log(x)$,..., $x^{\rho_i} \log(x)^{m_i - 1}$, $i = 1, ..., q$, *form a* $\mathbb{C}$*-basis of local solutions of* $L_0 y = 0$ *at* 0.

Proof. Clearly, these solutions are $\mathbb{C}$-linearly independent. As there cannot be more than $n = \operatorname{ord} L = \deg \chi_0 = \sum_{i=1}^{q} m_i$ linearly independent solutions, they already form a $\mathbb{C}$-basis. ↺

Our next objective will be to "lift" the solutions of Euler equations to solutions of arbitary linear differential equations $Ly = 0$ by interpreting the operator L as an arbitrarily small perturbation of its initial form $L_0 = \operatorname{in}(L)$. This cannot work without some extra effort since whereas the initial form L_0 of an operator L sends monomials x^i in $\mathbb{C}((x))$ to monomials $x^{i+\tau}$, this is no longer true for $\underline{L}_0$, since already for $\underline{L}_0 = \underline{\partial}$ we have that $\underline{L}_0(x^i z^k) = (iz + k) x^{i-1} z^{k-1}$ is now a binomial in z. We may have to apply L_0^{-1} to each degree in z, say, $L^{-1} : \mathbb{C}((x))[z]_k \to \mathbb{C}((x))[z]_k$, instead of trying with $\underline{L}_0^{-1}$. But, of course, L_0^{-1} is no longer compatible with the substitution $z \to \log(x)$. So complications will have to be expected - but we will resolve them.