FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART IV: NORMAL FORM THEOREM FOR MAXIMAL EXPONENTS

The situation we will consider in this section is as follows:

We denote by $\mathcal{O}$ the ring of germs at 0 of holomorphic functions on $\mathbb{C}$. It identifies with the ring $\mathbb{C}\{x\}$ of convergent power series. Its completion $\widehat{\mathcal{O}} = \mathbb{C}[[x]]$ consists of formal power series. We will treat both cases in parallel. Let $L = \sum_{j=0}^n \sum_{i=0}^\infty c_{ij} x^i \partial^j \in \mathcal{O}[\partial]$ be a linear differential operator with holomorphic coefficients. Decompose it into

$$L = L_0 + L_1 + L_2 + \cdots,$$

where L_i are Euler operators of increasing shifts $s_0 < s_1 < s_2 < \cdots$. Up to multiplication of L with a monomial in x we may and will assume that L_0 has shift 0, i.e., sends monomials x^k to $\chi(k)x^k$, where χ denotes the indicial polynomial of L_0 (or, of L) at 0. We call L_0 the initial form of L at 0. The roots ρ of χ in $\mathbb{C}$ are called the local exponents of L at 0, and their multiplicities are denoted by m_ρ.

We say that L has a $regular\ singularity$ at 0 if L_0 is an operator of the same order as L. It is equivalent to say that the coefficient c_{nn} of L is non-zero, or that $\sum_\rho m_\rho = n$, or that $L = \sum_{j=0}^n a_j(x)\partial^j$ has quotients a_i/a_n which a pole of order at most $n-i$ at 0. Fuchs' original definition of regular singularities was that $Ly = 0$ has a basis of $moderate$ solutions at 0. We will make this precise and prove the equivalence with the other definitions in a later section.

$Examples.$ (1) The equation $x^k y' + y = 0$ has a regular singularity at 0 if and only if $k \le 1$.

(2) The second order equation $x^k y'' + x^m y' + y = 0$ has a regular singularity at 0 if and only if $x^m/x^k = x^{m-k}$ has a pole of order $\le 2-1 = 1$, and $1/x^k = x-k$ has a pole of order $\le 2-0 = 2$ at 0. This is equivalent to $k \le 2$ and $k \le m+1$. In particular, if $k = 2$, then $m \ge 1$.

(3) Consider now $x^2y'' + 3x'y' + y - xy = 0$. The initial form at 0 is $L_0 = x^2\partial^2 + 3x\partial + 1$ with shift 0, while $L_1 = -x$, the multiplication with x, has shift $+1$. The indicial polynomial χ is $\rho(\rho-1) + 3\rho + 1 = (\rho+1)^2$, with root $\rho = -1$ of multiplicity $m_\rho = 2$. The associated Euler equation $L_0 y = 0$ has solutions $y_1 = x^{-1}$, $y_2 = x^{-1}\log(x)$. By the results of Fuchs-Thomé-Frobenius, the solutions of $Ly = 0$ are

$$y_1 = x^{-1}h_0(x), y_2 = x^{-1}h_1(x) + x^{-1}\log(x)h_0(x)$$

with holomorphic functions $h_0, h_1 \in \mathcal{O}$. We will prove this in the course of the classes in a modern and more conceptual language. Let us proceed step by step.

We start with the classical description of one specific local solution of a linear differential equation at a regular singular point, assuming an extra assumption on the involved local exponent ρ:

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Theorem. [Fuchs, Thomé, Frobenius] *Let* 0 *be a regular singularity of an* n*-th order linear differential equation* $Ly = 0$ *with holomorphic coefficients, and let* $\rho \in \mathbb{C}$ *be a local exponent of* L *at* 0*. Assume that* ρ *is a maximal local exponent of* L *modulo* $\mathbb{Z}$*, i.e., that* $\rho + k$ *is not a local exponent for any integer* $k \geq 1$*. Then there exists a holomorphic function* $h(x)$ *in the neighborhood of* 0 *such that* $y(x) = x^\rho \cdot h(x)$ *is a solution of* $Ly = 0$*.*

We will establish this result as a corollary of the normal form theorem to be proven below. It goes as follows:

Theorem. (Normal form theorem, single maximal exponent) *Let* $L \in \mathcal{O}[\partial]$ *be an* n*-th order linear differential operator with holomorphic coefficients. Let* $\rho \in \mathbb{C}$ *be a maximal local exponent of* L *at* 0 *modulo* $\mathbb{Z}$*, i.e., such that* $\rho + k$ *is not a local exponent for any positive integer* k*. Denote by* L_0 *the initial form of* L *at* 0*, and assume that* L_0 *has shift* 0*. Set* $\mathcal{F} = x^\rho \mathcal{O}$ *and* $\widehat{\mathcal{F}} = x^\rho \widehat{\mathcal{O}}$ *and write also* L *and* $\widehat{L}$ *for the linear maps on* $\mathcal{F}$ *and* $\widehat{\mathcal{F}}$ *induced by* L*. There exists a linear automorphism*

$$\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$$

such that on $\widehat{\mathcal{F}}$

$$\widehat{L} \circ \widehat{u}^{-1} = \widehat{L}_0.$$

If 0 *is a regular singular point of* L*, then* u *restricts to a linear automorphism*

$$u : \mathcal{F} \to \mathcal{F}$$

such that on $\mathcal{F}$

$$L \circ u^{-1} = L_0.$$

This result justifies the wording that L_0 is a *normal form* of L on $\mathcal{F}$. From this we immediately obtain

Corollary. *Let* $y_1 = x^\rho$ *be the first solution of the associated Euler equation* $L_0 y = 0$*. Then* $u^{-1}(y_1) = u^{-1}(x^\rho)$ *is a solution of* $Ly = 0$*.* $\circlearrowleft$

Remarks. (a) A suitable u is given as

$$u = \mathrm{Id}_{x^\rho \cdot \mathbb{C}\{x\}} - S \circ T,$$

where $S = L_{0|\mathcal{H}}^{-1}$ is the inverse of the restriction of L_0 to a direct complement $\mathcal{H}$ of its kernel in $\mathcal{F}$, and where $T = L_0 - L$ is the negative of the tail of L. Its inverse $v = u^{-1}$ is given as the geometric (or: von Neumann) series $v = \mathrm{Id}_\mathcal{F} + \sum_{k=1}^\infty (S \circ T)^k$, see the proof.

(b) In case that ρ has multiplicity $m_\rho > 1$, the other solutions $y_i = x^\rho \log(x)^{i-1}$ of $L_0 y = 0$ can also be lifted to solutions of $Ly = 0$, but this requires to introduce logarithms in the function space $\mathcal{F}$. See below.

(c) The regularity of the singularity of L is used only for the convergence part, i.e., that $\widehat{u}$ sends $\mathcal{F}$ into $\mathcal{F}$. Later on, for constructing a whole basis of solutions, it will be used again so as to have sufficiently local exponents, namely such that their multiplicities $\sum m_\rho = n$ sum up to n.

(d) The maximality of ρ with respect to $\mathbb{Z}$ among the local exponents of L is crucial. If this is not assumed, more complicated function spaces $\mathcal{F}$ have to be considered, both for the normal form theorem and the description of the solutions as in the corollary.

(e) The part for formal power series works for any field of characteristic 0. The case of positive characteristic is much more complicated and has been developed and proven recently by Florian Fürnsinn from the University of Vienna.

Theorem. (Normal form theorem, multiple maximal exponent) *Let $L \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients. Let $\rho \in \mathbb{C}$ be a maximal local exponent of L at 0 modulo $\mathbb{Z}$, i.e., such that $\rho + k$ is not a local exponent for any positive integer k. Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. Set $\mathcal{F} = x^\rho \mathcal{O}[z]$ and $\widehat{\mathcal{F}} = x^\rho \widehat{\mathcal{O}}[z]$. Denote by $\underline{\partial}$ the extension of ∂ to $\mathcal{O}[z]$ defined by $\underline{\partial} x = 1$, $\underline{\partial} z = x^{-1}$, and write accordingly $\underline{L} \in \mathcal{O}[\underline{\partial}]$ for the induced operator. There exists a linear automorphism $\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ such that on $\widehat{\mathcal{F}}$*

$$\underline{L} \circ \widehat{u}^{-1} = \underline{L}_0.$$

If 0 is a regular singular point of L, then u restricts to a linear automorphism $u : \mathcal{F} \to \mathcal{F}$ such that on $\mathcal{F}$

$$\underline{L} \circ u^{-1} = \underline{L}_0.$$

Again, we immediately obtain

Corollary. *Let $y_1 = x^\rho, ..., y_{m_\rho} = x^\rho \log(x)^{m_\rho - 1}$ be the solutions of the Euler equation $L_0 y = 0$. Then $u^{-1}(y_1) = u^{-1}(x^\rho), ..., u^{-1}(y_{m_\rho}) = u^{-1}(x^\rho \log(x)^{m_\rho - 1})$ are solutions of $Ly = 0$.* ◯

Remarks. (a) To get a basis of solutions of $Ly = 0$ one has to vary the local exponents ρ. But there might occur two obstructions: First, some local exponents may not be maximal, and then extra caution has to be taken; and, secondly, there the sum of the multiplicities m_ρ may be strictly less than n. In this case, as seen above, the singularity is not regular. There still exists a basis of solutions in a suitable function space. It will involve exponentials $\exp(P(1/x^q))$ of rational functions, P a polynomial, $q \in \mathbb{N}$ an integer. This is a classical theorem of Fabry from 1885. Nicholas Merkl from the University of Vienna is currently preparing a modern version of it along the lines of the normal form theorem.

(b) The special shape of the solutions $u^{-1}(y_i)$ of $Ly = 0$ as indicated in the theorem of Fuchs-Thomé-Frobenius above follows from the explicit description of a suitable automorphism u.

Proof. (a) We only prove here the normal form theorem in case where L acts on $\mathcal{F} = x^\rho \widehat{\mathcal{O}}$, respectively, $\mathcal{F} = x^\rho \mathcal{O}$. This will already reveal the technique and the various arguments. The case of $\underline{L}$ acting on $\mathcal{F} = x^\rho \mathcal{O}[z]$ goes along the same lines, and we will indicate the places where modifications have to be applied.

Write $L = L_0 - T$ with $-T = L_1 + L_2 + \cdots$ the tail of L. As T has positive shift (recall that L_0 is assumed to have shift 0), we have

$$T(x^\rho \mathcal{O}) \subset x^{\rho+1} \mathcal{O}.$$

This will be used in a moment. As L_0 annihilates x^ρ, we also get $L_0(x^\rho \mathcal{O}) \subset x^{\rho+1} \mathcal{O}$. But now, as ρ is maximal modulo $\mathbb{Z}$ among the local exponents of L, we know that $L_0(x^{\rho+k}) \neq 0$ for all $k \geq 1$. This implies that $L_0(\mathcal{F}) = x\mathcal{F}$. It hence contains the image of T. This is crucial for the argument to follow, and it also holds for $\underline{L}$ and $\mathcal{F} = x^\rho \mathcal{O}[z]$, as one checks with a little patience.

Now, as $L_0 : \mathcal{F} \to x\mathcal{F}$ is surjective, it induces a linear isomorphism when restricted to a direct complement $\mathcal{H}$ of the kernel $\mathrm{Ker}(L_0)$ of L_0 in $\mathcal{F}$,

$$L_{0|\mathcal{H}} : \mathcal{H} \to x\mathcal{F}.$$

Denote by $S = L_{0|\mathcal{H}}^{-1}$ its inverse,

$$S : x\mathcal{F} \to \mathcal{H}.$$

(b) At this point the proof splits into two case, the case of formal power series and the one with convergent series. Let us do first the formal case, $\widehat{O} = \mathbb{C}[[x]]$, and write $\widehat{u} : \widehat{\mathcal{H}} \to x\widehat{\mathcal{F}}$ for the map defined above (we still write L, S and T without "$\widehat{\ }$").

We claim that

$$\widehat{v} := \mathrm{Id}_{\widehat{\mathcal{F}}} + \textstyle\sum_{k=1}^{\infty}(S \circ T)^k : \mathcal{F} \to \mathcal{F}$$

is well defined and an inverse to $\widehat{u}$. To see this, juste recall that T, when applied to a power seres, increases its order at least by 1. And S preserves the order, since L_0 has shift 0. Actually, one may choose for S the map (the "integration operator") defined by

$$S(x^{\rho+k}) = \frac{1}{\chi(\rho+k)} \cdot x^{\rho+k}.$$

So $S \circ T$ maps $x^m\mathcal{F}$ into $x^{m+1}\mathcal{F}$. But as $\mathbb{C}[[x]]$ is complete with respect to the x-adic topology (with neighborhoods of 0 given by the powers $(x)^m$ of the maximal ideal (x)), we can conclude that $\widehat{v}$ defines indeed a map from $\mathcal{F}$ to $\mathcal{F}$. And clearly, $\widehat{v}$ is then an inverse to $\widehat{u}$, all maps being linear.

It remains to prove that $L_{0|\mathcal{F}} \circ u = L_{|\mathcal{F}}$, where we write subscripts to emphasize that we mean the linear maps on $\mathcal{F}$ and not the abstract operators. It is also helpful to convince oneself that in the equations below all computations are valid transformations. The proof of the equality is now easy (and nice). Namely, we have

$$\begin{aligned}
L_{0|\mathcal{F}} \circ u &= L_{0|\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{F}} - S \circ T) \\
&= L_{0|\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{F}} - S \circ (L_0 - L)_{|\mathcal{F}}) \\
&= L_{0|\mathcal{F}} \circ (\mathrm{Id}_{\mathcal{F}} - S \circ L_{0|\mathcal{F}} + S \circ L_{|\mathcal{F}}) \\
&= L_{0|\mathcal{F}} - L_{0|\mathcal{F}} \circ S \circ L_{0|\mathcal{F}} + L_{0|\mathcal{F}} \circ S \circ L_{|\mathcal{F}} \\
&= L_{0|\mathcal{F}} \circ S \circ L_{|\mathcal{F}} \\
&= L_{|\mathcal{F}},
\end{aligned}$$

using twice that S is an inverse to $L_{0|\mathcal{H}}$ and hence $L_{0|\mathcal{F}} \circ S = \mathrm{Id}_{x\mathcal{F}}$. And recall that L maps $\mathcal{F}$ into $x\mathcal{F}$, so all compositions are well defined. This concludes the proof of the theorem for formal power series.

It is not hard to see that the same reasoning applies to the extension $\mathcal{F} = x^\rho\mathcal{O}[z]$ and the associated linear map $\underline{L}_{|\mathcal{F}}$. We leave this as an exercise.

(c) As for the convergent case, one has to look a bit carefully what the maps T and S do on power series with prescribed radius of convergence. Let $s > 0$ be a small real number and denote by $\mathcal{O}_s$ the ring of power series $h = \sum_{k=0}^{\infty} a_k x^k$ such that $|h|_s := \sum_{k=0}^{\infty} |a_k| s^k < \infty$. This is a Banach space, and $| - |_s$ is a norm on it. We thus get the induced Banach space $\mathcal{F}_s$ and linear maps $\mathcal{F}_s \to \mathcal{F}_s$ can be equipped with the operator norm (in the sense of functional analysis), denoted by $|| - ||_s$. We may choose $s > 0$ sufficiently small such that $L : \mathcal{F}_s \to \mathcal{F}_s$ is well defined (recall that $L \in \mathcal{O}[\partial]$ has finitely many convergent coefficients, hence belongs to $\mathcal{O}_s[\partial]$ for s small).

We show that $||(S \circ T)||_s < 1$. This will imply the convergence of the sum $\sum_{k=0}^{\infty}(S \circ T)^k$ defining v as a map on $\mathcal{F}_s$. To this end, we show that there is a constant $0 < C < 1$ such that

$$|S(T(x^\rho h))|_s \leq C \cdot |x^\rho h|_s$$

for all $h \in \mathcal{O}_s$. The formulas are slightly complicated. For $h = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{O}$ we have

$$T(x^\rho h) = -\sum_{i-j>0} \sum_{k=0}^{\infty} (\rho+k)^{\underline{j}} c_{ij} a_k x^{\rho+k+i-j},$$

and

$$S(T(x^\rho h)) = -\sum_{i-j>0} \sum_{k=0}^{\infty} \frac{(\rho+k)^{\underline{j}}}{\chi(\rho+k+i-j)} c_{ij} a_k x^{\rho+k+i-j}.$$

As $i - j > 0$ and $k \geq 0$, no $\rho + k + i - j$ appearing in the denominator is a root of the indicial polynomial χ. Hence the ratio

$$\frac{(\rho+k)^{\underline{j}}}{\chi(\rho+k+i-j)} = \frac{(\rho+k)^{\underline{j}}}{\sum_{\ell-m=0} c_{\ell m}(\rho+k+i-j)^{\underline{m}}} = \frac{(\rho+k)^{\underline{j}}}{\sum_{m=0}^{n} c_{mm}(\rho+k+i-j)^{\underline{m}}}$$

is well defined. But, as the singularity of L is regular, the order of L_0 is n and hence $c_{nn} \neq 0$. This implies that $(\rho + k + i - j)^{\underline{n}}$ appears in the denominator with non-zero coefficient. It is at that place where the regularity of the singularity is used in the proof. Note that in the ratio some i could be less than j and hence the respective $\rho + k + i - j$ would be smaller than the $\rho + k$ in the numerator. But the bound $j \leq n$ for j nevertheless ensures that the ratio remains bounded, say $\leq c$, in absolute value as k tends to ∞. Taking norms on both sides of the above equality for $S(T(x^\rho h))$ yields, for $s \leq 1$, the estimate

$$|S(T(x^\rho h))|_s \leq c \sum_{i-j>0} \sum_{k=0}^{\infty} |c_{ij}||a_k| s^{\rho+k+i-j} = c \sum_{i-j>0} |c_{ij}| s^{i-(j)} \sum_{k=0}^{\infty} |a_k| s^{\rho+k}.$$

But by assumption the coefficients $\sum_{i=0}^{\infty} c_{ij} x^i$ of L belong to $\mathcal{O}_s$ for all $j = 0, ..., n$. This implies in particular $\sum_{i>j}^{\infty} c_{ij} x^i \in \mathcal{O}_s$ and then, after division by x^{j+1} and since $i \geq j + 1$, that

$$\textstyle\sum_{i>j}^{\infty} c_{ij} x^{i-(j+1)} \in \mathcal{O}_s.$$

We get that

$$\textstyle\sum_{i-j>0} |c_{ij}| s^{i-(j)} = s \cdot \sum_{i-j>0} |c_{ij}| s^{i-(j+1)} \leq c' s$$

for some $c' > 0$ independent of s. This inequality allows us to bound $|S(T(x^\rho h))|_s$ from above by

$$|S(T(x^\rho h))|_s \leq cc' s \sum_{k=0}^{\infty} |a_k| s^{\rho+k} = cc' s |x^\rho h|_s.$$

Take now $0 < s_0$ sufficiently small with $s_0 < \frac{1}{cc'}$, and get a constant $0 < C < 1$ such that for $0 < s \leq s_0$ one has

$$|S(T(x^\rho h))|_s \leq C \cdot |x^\rho h|_s.$$

This shows that $||(S \circ T)_s|| < 1$ holds on F_s for $0 < s \leq s_0$ as required. The convergence proof is completed. $\circlearrowleft$

Remark. The convergence proof extends to $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ (polynomials in z of degree $< m$) since this is a finite free module over $\mathcal{O}$. As the action of $\underline{L}$ on this extended $\mathcal{F}$ does not increase the degree in z, the restriction to this module is justified.