FUCHSIAN DIFFERENTIAL EQUATIONS: NOTES FALL 2022

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NOTES PART V: NORMAL FORM THEOREM WITH LOGARITHMS

Let us recall from last time the second, more general version of the normal form theorem where the local exponent ρ has multiplicity $m = m_\rho \geq 1$ and where we considered the operator L as acting on the enlarged function space $\mathcal{F} = x^\rho \mathcal{O}[z]$. In fact, we can (and will) even restrict to the smaller space $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ of polynomials in z of degree $< m$. Normalizing the action of the operator L on this space will be sufficient and perfectly suited to construct m solutions $y_i = x^\rho \log(x)^i h_i(x)$ of $Ly = 0$, with $0 \leq i < m$ and $h_i \in \mathcal{O}$ holomorphic. We will still have to assume that ρ is maximal with respect to $\mathbb{Z}$. The case where ρ is no longer maximal requires extra constructions and will be treated next time.

Theorem. (Normal form theorem with logarithms) *Let $L = \sum_{j=0}^n p_j(x)\partial^j \in \mathcal{O}[\partial]$ be an n-th order linear differential operator with holomorphic coefficients p_j in $\mathcal{O}$. Let $\rho \in \mathbb{C}$ be a maximal local exponent of L at 0 modulo $\mathbb{Z}$, i.e., $\rho + k$ is not a local exponent for any positive integer k. Let $m = m_\rho \geq 1$ be its multiplicity as a root of the indicial polynomial χ of L. Denote by L_0 the initial form of L at 0, and assume that L_0 has shift 0. Set $\mathcal{F} = x^\rho \mathcal{O}[z]_{<m}$ and $\widehat{\mathcal{F}} = x^\rho \widehat{\mathcal{O}}[z]_{<m}$. Denote by $\underline{\partial}$ the extension of ∂ to $\mathcal{O}[z]$ defined by $\underline{\partial} x = 1$, $\underline{\partial} z = x^{-1}$, and write accordingly $\underline{L} = \sum_{j=0}^n p_j \underline{\partial}^j \in \mathcal{O}[\underline{\partial}]$ for the induced operator. There exists a linear automorphism $\widehat{u} : \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ such that the linear maps on $\widehat{\mathcal{F}}$ induced by $\underline{L}$ and $\underline{L}_0$ and denoted by the same letters satisfy*

$$\underline{L} \circ \widehat{u}^{-1} = \underline{L}_0.$$

Moreover, if 0 is a regular singular point of L, then $\widehat{u}$ restricts to a linear automorphism $u : \mathcal{F} \to \mathcal{F}$ such that the linear maps on $\mathcal{F}$ induced by $\underline{L}$ and $\underline{L}_0$ satisfy

$$\underline{L} \circ u^{-1} = \underline{L}_0.$$

Remarks. (a) The automorphism $\widehat{u}$ is again of the form $\widehat{u} = \mathrm{Id}_{\widehat{\mathcal{F}}} - S \circ T$ with $T = L_0 - L$ and S the inverse of the restriction $L_{0|\widehat{\mathcal{H}}}$ of L_0 to a direct complement $\widehat{\mathcal{H}}$ of its kernel in $\widehat{\mathcal{F}}$ as in the first version of the normal form theorem. Accordingly, u has the form $u = \mathrm{Id}_{\mathcal{F}} - S \circ T$.

(b) We do not allow L to have coefficients depending also on the variable z, i.e., lying in $\mathcal{O}[z]$. This would correspond to differential equations whose coefficients involve powers of logarithms. It is not clear whether this case would have interesting applications.

(c) The convergence part of the theorem requires again that 0 is a regular singularity of L. The proof is analogous to the case without logarithms, using the same estimates.

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Before proving the theorem let us state immediately its central output about the solutions of $Ly = 0$:

Corollary. *Let $y_1 = x^\rho, ..., y_m = x^\rho \log(x)^{m-1}$ be the solutions of the Euler equation $L_0 y = 0$. Then $u^{-1}(y_1) = u^{-1}(x^\rho), ..., u^{-1}(y_m) = u^{-1}(x^\rho \log(x)^{m-1})$ are solutions of $Ly = 0$. More explicitly, these solutions are of the form, for $1 \leq i \leq m$,*

$$y_1(x) = x^\rho h_1(x),$$

$$y_2(x) = x^\rho [h_2(x) + h_1(x) \log(x)],$$

$$y_i(x) = x^\rho [h_i(x) + h_{i-1}(x) \log(x) + \ldots + h_1(x) \log(x)^{i-1}],$$

with $h_1, ..., h_m$ formal power series in $\widehat{O}$, respectively, holomorphic functions in $\mathcal{O}$.

Remark. The special shape of the solutions $y_i(x)$ stems from the explicit description of the normalizing automorphism u as given above, see the proof of the theorem.

Examples. The proof of the theorem will use the three lemmata 1, 2, 3 of part III (the section on Euler operators) describing the extensions $\underline{\partial}^j$ and $\underline{L}$ of derivations and differential operators to $x^\rho \mathcal{O}[z]$. To get a more concrete hold on these, let us consider two examples.

(1) Let $E = x^2\partial^2 - 3x\partial + 3$ be an Euler operator with indicial polynomial $\chi(t) = (t+1)^2$ and local exponent $\rho = -1$ of multiplicity $m = 2$. Let it act on $x^{-1}\mathcal{O}[z]_{<2}$. Then

$$\underline{E}(x^k z^i) = x^k[(k+1)^2 z^i + 2(k+1)iz^{i-1} + 2i(i-1)z^{i-2}].$$

We get $\mathrm{Ker}(E) = \mathbb{C}x^{-1} \oplus \mathbb{C}x^{-1}z$, and $\mathrm{Im}(E) = xx^{-1}\mathcal{O}[z]_{<2} = \mathcal{O}[z]_{<2}$.

(2) Let $E = x^3\partial^3 - 4x^2\partial^2 + 9x\partial - 9$ be an Euler operator with indicial polynomial $\chi(t) = (t-1)(t-3)^2$ and local exponents $\rho = 3$ of multiplicity $m = 2$ and $\sigma = 1$ of multiplicity 1. Let it act on $x\mathcal{O}[z]_{<2}$. Then

$$\underline{E}(x^k z^i) = x^k[(k-1)(k-3)^2 z^i + (3k-5)(k-1)iz^{i-1} + (6k-14)i^{\underline{2}}z^{i-2} + 6i^{\underline{3}}z^{i-3}].$$

The kernel is $\mathrm{Ker}(E) = \mathbb{C}x \oplus \mathbb{C}x^3 \oplus \mathbb{C}x^3 z$. Determine the image $\mathrm{Im}(E)$!

(3) Let finally $E = x^2\partial^2 - x\partial$ be with $\chi(t) = t(t-2)$ and local exponents $\rho = 2$ and $\sigma = 0$, both of multiplicity 1. Let it act on $\mathcal{F} = \mathcal{O} + x^2\mathcal{O} = \mathcal{O}$ since no logarithms are to be expected. Then

$$\underline{E}(x^k) = k(k-2)x^k$$

and hence $\mathrm{Ker}(E) = \mathbb{C} \oplus \mathbb{C}x^2$. The image is $\mathrm{Im}(E) = \mathbb{C}x + \mathcal{O}x^3 \subset \mathcal{O}$, which is now strictly contained in $x\mathcal{F} = x\mathcal{O}$. The "gap" occurs at x^2, and this will cause serious problems when trying to apply the arguments of the proof of the normal form theorem - recall that it relied heavily on the equality $\underline{L}_0(\mathcal{F}) = x\mathcal{F}$, and this fails in the present example. The reason is that there is *resonance* between the two local exponents, say, $\rho - \sigma \in \mathbb{Z}$. We will show in part VI of the notes how to overcome this problem.

Proof. Recall from Lemma 1, part III, the formula

$$\underline{\partial}^j = \partial^j + (\partial^j)'\partial_z + \tfrac{1}{2}(\partial^j)''\partial_z^2 + \ldots + \tfrac{1}{\ell!}(\partial^j)^{(\ell)}\partial_z^\ell + \ldots + \tfrac{1}{j!}(\partial^j)^{(j)}\partial_z^j,$$

where the derivatives $(\partial^j)^{(\ell)}$ are defined on $\mathcal{O}$ by $(\partial^j)^{(\ell)}(x^t) = (t^{\underline{j}})^{(\ell)}x^{t-j}$ while leaving z invariant. For an operator $L = \sum_{j=0}^n p_j\partial^j \in \mathcal{O}[\partial] \in \mathcal{O}[\partial]$ define accordingly its ℓ-th derivative as

$$L^{(\ell)} = \textstyle\sum_{j=0}^n p_j(\partial^j)^{(\ell)},$$

acting again on $\mathcal{O}$ while leaving z invariant. More explicitly,

$$(L^{(\ell)}\partial_z^\ell)(x^k z^i) = L^{(\ell)}(x^k)\cdot \partial_z^\ell(z^i) = (k^{\underline{j}})^{(\ell)} i^{\underline{\ell}} \cdot x^{k-j} z^{i-\ell}.$$

Then the preceding formula extends for the action $\underline{L}$ of L on $x^\rho\mathcal{O}[z]$ by linearity to

Lemma 4. *In the above situation, one has*

$$\underline{L} = L + L'\partial_z + \tfrac{1}{2}L''\partial_z^2 + \ldots + \tfrac{1}{\ell!}L^{(\ell)}\partial_z^\ell + \ldots + \tfrac{1}{n!}L^{(n)}\partial_z^n.$$

We will call this decomposition the *Taylor expansion* of $\underline{L}$ on $x^\rho\mathcal{O}[z]$.

Let us now turn to the actual proof of the normal form theorem with logarithms. The formula in Lemma 4 applies in particular to the initial form L_0 of L at 0. The key step is then, taking as function space on which L and L_0 act the space $\mathcal{F} = x^\rho\mathcal{O}[z]_{<m}$, where $m = m_\rho$ is again the multiplicity of the local exponent ρ of L, the following

Claim. *The image of $\underline{L}_0$ acting on $\mathcal{F}$ is $x\mathcal{F}$,*

$$\underline{L}_0(\mathcal{F}) = x\mathcal{F}.$$

Proof. The inclusion $\underline{L}_0(\mathcal{F}) \subset x\mathcal{F}$ is straightforward. Indeed, if $i < m$ then $i^{\underline{\ell}} = 0$ for $\ell \geq m \geq i+1$. Therefore, the formula of Lemma 3,

$$\underline{L}_0(x^\rho z^i) = x^\rho \cdot [\chi(\rho)z^i + \chi'(\rho)iz^{i-1} + \ldots + \tfrac{1}{n!}\chi^{(n)}(\rho)i^{\underline{n}}z^{i-n}],$$

reduces to

$$\underline{L}_0(x^\rho z^i) = x^\rho \cdot [\chi(\rho)z^i + \chi'(\rho)iz^{i-1} + \ldots + \tfrac{1}{(m-1)!}\chi^{(m-1)}(\rho)i^{\underline{m-1}}z^{i-m+1}].$$

This implies that $\underline{L}_0(x^\rho z^i) = 0$ for $i < m$, say $\underline{L}_0(x^\rho\mathbb{C}[z]_{<m} = 0$. Thus $\underline{L}_0(\mathcal{F}) \subset x\mathcal{F}$.

For the converse inclusion $\underline{L}_0(\mathcal{F}) \supset x\mathcal{F}$ we have to show that $x^{\rho+k}z^i \in \underline{L}_0(\mathcal{F})$ for all $k \geq 1$ and all $0 \leq i < m$. This is immediate if $i = 0$: then

$$\underline{L}_0(x^{\rho+k}) = L_0(x^{\rho+k}) = \chi(\rho+k)x^{\rho+k}$$

and $\chi(\rho+k) \neq 0$ since ρ is maximal modulo $\mathbb{Z}$. So $x^{\rho+k} \in \underline{L}_0(\mathcal{F})$ for all $k \geq 1$. Let now $i > 0$. We apply induction on i. By Lemma 4 we know that

$$\underline{L}_0(x^{\rho+k}z^i) = L_0(x^{\rho+k}z^i) + \sum_{\ell=1}^n \frac{1}{\ell!}L_0^{(\ell)}\partial_z^\ell(x^{\rho+k}z^i).$$

In terms of the derivatives of the indicial polynomial χ this reads as

$$\underline{L}_0(x^{\rho+k}z^i) = \chi(\rho+k)x^{\rho+k}z^i + \sum_{\ell=1}^n \frac{1}{\ell!}\chi^{(\ell)}(\rho+k)\frac{1}{\ell!}i^{\underline{\ell}}\cdot x^{\rho+k}z^{i-\ell}.$$

The first summand is non-zero as before, and the polynomial in z defined by the sum of the second summand has degree $< i$, hence belongs by induction on i to $\underline{L}_0(\mathcal{F})$. Therefore $x^{\rho+k}z^i \in \underline{L}_0(\mathcal{F})$ and the converse inclusion $\underline{L}_0(\mathcal{F}) \supset x\mathcal{F}$ is shown.

From this point on, the proof follows exactly the proof of the normal form theorem from part IV of the notes. The only thing to remark is that, for the convergence proof, one uses the fact that $\mathcal{F} = x^\rho\mathcal{O}[z]_{<m}$ is a *finite* free $\mathcal{O}$-module and so the Banach space argument applies again. $\circlearrowleft$