Noncommutative identities

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to Don Zagier, on the occasion of his $3{\cdot}4{\cdot}5$ *birthday, with love and admiration*

1 “Characteristic polynomial”

Let us fix an integer $n \geq 1$ and consider the algebra

$$\mathcal{A} := \mathbb{C}\langle X_1^{\pm 1}, \ldots, X_n^{\pm 1}\rangle$$

of noncommutative Laurent polynomials with coefficients in $\mathbb{C}$ in n invertible variables, i.e. the group ring of the free group Free_n on n generators:

$$\mathcal{A} = \mathbb{C}[\mathrm{Free}_n] = \Big\{ \sum_{g \in \mathrm{Free}_n} c_g \cdot g \,|\, c_g \in \mathbb{C},\ c_g = 0 \text{ for almost all } g \in \mathrm{Free}_n \Big\} .$$

Define a linear functional “Tr” on $\mathcal{A}$ by taking the constant term,

$$\text{““Tr''} : \mathcal{A} \to \mathbb{C}\,, \qquad \sum_g c_g \cdot g \mapsto c_{id} .$$

This functional vanishes on commutators, like the trace for matrices. By the analogy with the matrix case, we define the “characteristic polynomial”[1] for any $a \in \mathcal{A}$ as a formal power series in one (central) variable t:

$$P_a = P_a(t) := \text{“det''}(1 - ta) := \exp\left(- \sum_{k \geq 1} \frac{\text{“Tr''}(a^k)}{k} t^k \right) = 1 + \cdots \in \mathbb{C}[[t]] .$$

Theorem 1. *For any $a \in \mathcal{A}$ the series P_a is algebraic, i.e.*

$$P_a \in \overline{\mathbb{C}(t)} \cap \mathbb{C}[[t]] \subset \overline{\mathbb{C}((t))} .$$

[1] This is an algebraic analogue of so-called Fuglede-Kadison determinant, see [9].

Here are few examples:

- the case $n = 1$ is elementary, follows easily from the residue formula,
- for any $n \geq 1$ and

$$a = X_1 + X_1^{-1} + \cdots + X_n + X_n^{-1}$$

one can show that

$$P_a = \frac{\left(\frac{f+1}{2}\right)^n}{\left(\frac{nf+n-1}{2n-1}\right)^{n-1}}\,, \quad f = f(t) := \sqrt{1 - 4(2n-1)t^2} = 1 + \cdots \in \mathbb{Z}[[t]]\,.$$

- if $a = X_1 + \cdots + X_n + (X_1 \ldots X_n)^{-1}$ then the series P_a is an algebraic hypergeometric function.

A sketch of the proof:

Let us assume for simplicity that $a \in \mathbb{Z}[\mathrm{Free}_n]$, the general case is just slightly more complicated.

Step 1. For $a = \sum_g c_g \cdot g \in \mathbb{Z}[\mathrm{Free}_n]$ the series P_a also has integer coefficients. Indeed, it is easy to see that

$$P_a = \prod_{k \geq 1} \prod_{(g_1, \ldots, g_k)} (1 - c_{g_1} \ldots c_{g_k} t^k)$$

where for any $k \geq 1$ we take the product over sequences of elements of Free_n such that $g_1 \ldots g_k = id$ and the sequence $(g_1, \ldots, g_k)$ is strictly smaller than all its cyclic permutations for the lexicographic order on sequences associated with some total ordering of Free_n considered as a countable set. Similar argument works if we replace Free_n by an arbitrary torsion-free group.

Step 2. Consider the following series

$$F_a = F_a(t) := \sum_{k \geq 1} \text{“Tr”}(a^k)\, t^k \in \mathbb{Z}[[t]].$$

Then F_a is algebraic. This fact was rediscovered many times, see e.g. [2]. It follows from the theory of noncommutative algebraic series developed by N. Chomsky and M.-P. Schützenberger in 1963 (see [4]). We recommend to the reader to consult chapter 6 in [10], the algebraicity of the series F_a is the statement of Corollary 6.7.2 in this book.

Step 3. Recall the Grothendieck conjecture on algebraicity. It says that for any algebraic vector bundle with flat connection over an algebraic variety defined over a number field, all solutions of the corresponding holonomic system of differential equations are algebraic if and only if the p-curvature vanishes for all sufficiently large primes $p \gg 1$. There is a simple sufficient criterion for such a vanishing. Namely, it is enough to assume that there exists a fundamental system of solutions in formal power series at some algebraic point, such that all Taylor coefficients (in some local algebraic coordinate system) of all solutions have in total only finitely many primes in denominators.

The Grothendieck conjecture in its full generality is largely inaccessible by now. The only two general results are a theorem by N. Katz on the validity of the Grothendieck conjecture for Gauss-Manin connections, and a theorem of D. Chudnovsky and G. Chudnovsky [5] in the case of line bundles over algebraic curves (see also Exercise 5 to §4 of Chapter VIII, page 160 in [1]). This is exactly our case, by the previous step, and because

$$\frac{d}{dt} P_a = -\frac{F_a}{t} P_a \, .$$

$\square$

2 Noncommutative integrability, the case of two variables

Let now consider the algebra of noncommutative polynomials in two (non-invertible) variables

$$\mathcal{A} = \mathbb{C}\langle X, Y \rangle \ .$$

For any integer $d \geq 1$ we consider the variety $\mathcal{M}_d$ of $\mathrm{GL}_d(\mathbb{C})$-equivalence classes (by conjugation) of d-dimensional representations $\rho : \mathcal{A} \to \mathrm{Mat}_{d\times d}(\mathbb{C})$ of $\mathcal{A}$. More precisely, we are interested only in generic pairs of matrices $(\rho(X), \rho(Y))$ and treat variety $\mathcal{M}_d$ birationally. It has dimension $d^2 + 1$.

For generic ρ we consider the "bi-characteristic polynomial" in two commutative variables

$$P_\rho = P_\rho(x, y) := \det(1 - x\rho(X) - y\rho(Y)) = 1 + \cdots \in \mathbb{C}[x, y] \ .$$

The equation $P_\rho(x, y) = 0$ of degree $\leq d$ defines so-called Vinnikov curve $\mathcal{C}_\rho \subset \mathbb{C}P^2$. The number of parameters for the polynomial P_ρ is $\frac{(d+1)(d+2)}{2} - 1$, and it is strictly smaller than $\dim \mathcal{M}_d$ for $d \geq 3$. The missing parameters correspond to the natural line bundle $\mathcal{L}_\rho$ on $\mathcal{C}_\rho$ (well-defined for generic ρ) given by the kernel of operator $(1 - x\rho(X) - y\rho(Y))$ for $(x, y) \in \mathcal{C}_\rho$. Bundle $\mathcal{L}_\rho$

has the same degree as a square root of the canonical class of $\mathcal{C}_\rho$, and defines a point in a torsor over the Jacobian $\mathrm{Jac}(\mathcal{C}_\rho)$. For any given generic curve $\mathcal{C} \subset \mathbb{C}P^2$ of degree d line bundles on $\mathcal{C}$ depend on $\mathrm{genus}(\mathcal{C}) = \frac{(d-1)(d-2)}{2}$ parameters. Now the dimensions match:

$$\dim \mathcal{M}_d = d^2 + 1 = \frac{(d+1)(d+2)}{2} - 1 + \frac{(d-1)(d-2)}{2} \, .$$

The conclusion is that $\mathcal{M}_d$ is fibered over the space of planar curves of degree d, with the generic fiber being a torsor over an abelian variety. Hence we have one of simplest examples of an *integrable system*. Any integrable system has a commutative group of discrete symmetries, i.e. birational automorphisms preserving the structure of the fibration, identical on the base, and acting by shifts on the generic fiber. Similarly, one can consider an abelian Lie algebra consisting of rational vertical vector fields which are infinitesimal generators of shifts on fibers.

Now I want to consider *noncommutative* symmetries, i.e. certain universal expressions in free variables (X, Y) which can be specialized and make sense for any $d \geq 1$. An universal discrete symmetry is an automorphism of $\mathcal{A}$ (or maybe of some completion of $\mathcal{A}$) which preserves the conjugacy class of any linear combination $Z(t) := X + tY$, $t \in \mathbb{C}$. Indeed, in this case for any $d \geq 1$ and any representation, the value of the bi-characteristic polynomial P_ρ at any pair of complex numbers $(x, y) \in \mathbb{C}^2$ is preserved, as it can be written as $\det(1 - x\rho(Z(y/x)))$. Hence the automorphism under consideration is inner on both variables X and Y:

$$X \mapsto R \cdot X \cdot R^{-1}, \;\; Y \mapsto R' \cdot Y \cdot (R')^{-1} \, .$$

We are interested in automorphisms of $\mathcal{A}$ only up to inner automorphisms, therefore we may safely assume that $R' = 1$. Thus, the question is reducing to the following one:

$$\boxed{\begin{array}{c} \textit{find } R \textit{ such that for any } t \in \mathbb{C} \textit{ there exists } R_t \textit{ such that} \\ R \cdot X \cdot R^{-1} + tY = R_t \cdot (X + tY) \cdot R_t^{-1} \end{array}}$$

First, let us make calculations on the Lie level. Denote by $\mathfrak{g}$ the Lie algebra of derivations δ of $\mathcal{A}$ of the form

$$\delta(X) = [D, X] \text{ for some } X \in \mathcal{A}, \quad \delta(Y) = 0$$

and such that for any $t \in \mathbb{C}$ there exists $D_t \in \mathcal{A}$ such that

$$\delta(X + tY) = [D_t, X + tY] \iff [D, X] = [D_t, X + tY] \, .$$

It is easy to classify such derivations, and one can check that the following elements form a linear basis of $\mathfrak{g}$:

$$\delta_{n,m}(X) = [c_{n,m}, X],\ \ \delta_{n,m}(Y) = 0,\ \ n \geq 0,\ m \geq 1$$

where for any $n, m \geq 0$ we define

$$c_{n,m} := \sum_{\frac{(n+m)!}{n!m!}\ \text{shuffles}\ w} w\ ,$$

i.e. the sum of all words in X, Y containing n letters X and m letters Y. Elements $D_t \in \mathcal{A}$ corresponding to the derivation $\delta_{n,m}$ are given by

$$D_t = \sum_{0 \leq k \leq n} c_{n-k,m+k}\, t^k\ .$$

A direct calculation shows that $\mathfrak{g}$ is an abelian Lie algebra.

Let us go now the completions of algebra $\mathcal{A}$, and of Lie algebra $\mathfrak{g}$:

$$\widehat{\mathcal{A}} := \mathbb{C}\langle\langle X, Y\rangle\rangle,\ \ \widehat{\mathfrak{g}} := \prod_{n\geq 0, m\geq 1} \mathbb{C}\cdot\delta_{n,m}\ .$$

Then the action of $\widehat{\mathfrak{g}}$ on $\widehat{\mathcal{A}}$ exponentiates a continuous group action

$$\widehat{\mathfrak{g}} \overset{\exp}{\simeq} \widehat{G} \subset \mathrm{Aut}(\widehat{\mathcal{A}})\ .$$

For any $\delta \in \widehat{\mathfrak{g}}$ the corresponding one-parameter group of automorphisms acts by

$$\exp(\tau\cdot\delta) : X \mapsto R(\tau)\cdot X\cdot R(\tau)^{-1},\ \ Y \mapsto Y \qquad \forall\, \tau \in \mathbb{C}$$

for certain invertible element $R(\tau) \in \widehat{\mathcal{A}}^{\times}$. An easy calculation shows that $R(\tau)$ is the unique solution of the differential equation

$$\frac{d}{d\tau}R(\tau) = \delta(R(\tau)) + R(\tau)\cdot D, \qquad R(0) = 1$$

where $D \in \widehat{\mathcal{A}}$ is such that $\delta(X) = [D, X]$. The value $R(\tau)_{|\tau=1}$ gives $\exp(\delta)$.

Now we can start to look for a class of elements $\delta \in \widehat{\mathfrak{g}}$ such that the corresponding automorphism $\exp(\delta)$ is sufficiently nice, e.g. if it makes some sense for $\mathcal{A}$ without passing to the completion.

Let us encode a generic element δ as before by the corresponding generating series in *commutative* variables x, y:

$$\delta = \sum_{n,m} f_{n,m}\delta_{n,m} \in \widehat{\mathfrak{g}} \;\rightsquigarrow\; \widetilde{\delta} := \sum_{n,m} f_{n,m} x^n y^m \in \mathbb{C}[[x,y]] \ .$$

I suggest the following Ansatz:

$$\boxed{\widetilde{\delta} \textit{ is the logarithm of a rational function in } x, y.}$$

Hypothetically, for such δ the corresponding automorphism $\exp(\delta)$ of $\widehat{\mathcal{A}}$ can be extended to certain "algebraic extension" of $\mathcal{A}$. A good indication is

Theorem 2. *For any $P = P(x,y) = 1 + \cdots \in \mathbb{C}[x,y]$ expand*

$$\log(P) = \sum_{n,m} f_{n,m} x^n y^m \in \mathbb{C}[[x,y]] \ .$$

Then the series

$$\exp\left(\sum_{n,m} \frac{(n+m)!}{n!m!} f_{n,m} x^n y^m\right)$$

is algebraic.

This result is elementary, and I leave it as an exercise to the reader. (Hint: use the residue formula twice.) It implies that the image of R under the abelianization morphism

$$\mathbb{C}\langle\langle X, Y\rangle\rangle \twoheadrightarrow \mathbb{C}[[x,y]], \;\; X \mapsto x,\, Y \mapsto y$$

is algebraic.

Example. Consider the case

$$\widetilde{\delta} = \log(1 - xy) = -\sum_{k\geq 1} \frac{x^k y^k}{k} \ .$$

Then one can show that

$$R = 1 - YX - C \in \widehat{\mathcal{A}}^{\times}$$

where C is the unique solution of the equation

$$C = X \cdot (1 - C)^{-1} \cdot Y \ .$$

It can be written

$$C = XY + XXYY + XXYXYY + XXXYYY + \cdots = () + (()) + (()()) + ((())) + \dots$$

as the sum of all irreducible bracketings if we replace X by (and Y by).

The equation for C is equivalent to the generic "quadratic equation"

$$T^2 + AT + B = 0$$

by the substitutions

$$A := X^{-1}, \quad B := -X^{-1}Y, \quad T := X^{-1}C \; .$$

The invertible elements $R_t \in \widehat{\mathcal{A}}^{\times}, \; t \in \mathbb{C}$ are given by

$$R_t := R \cdot (1 - tT^2), \quad R \cdot X \cdot R^{-1} + tY = R_t \cdot (X + tY) \cdot R_t^{-1} \; .$$

I'll finish with another example of an integrable system. Few years ago together with S. Duzhin we discovered numerically that the rational map

$$S_{-1} : (X, Y) \mapsto (XYX^{-1}, \, (1 + Y^{-1})\, X^{-1})$$

should be a discrete symmetry of an integrable system, where X, Y are two $d \times d$ matrices for $d \geq 1$. Recently O. Efimovskaya and Th. Wolf found an explanation (see [7]). Namely, their results suggest that the conjugacy class of the Lax operator which is the matrix $L(t)$ of size $2d \times 2d$, defined as

$$L(t) := \begin{pmatrix} Y^{-1} + X & tY + Y^{-1}X^{-1} + X^{-1} + 1 \\ Y^{-1} + \frac{1}{t}X & Y + Y^{-1}X^{-1} + X^{-1} + \frac{1}{t} \end{pmatrix}$$

does not change under the discrete symmetry S_{-1} as above, for any $t \in \mathbb{C}$. Indeed, one can check directly that $S_{-1}(L(t)) = V(t)L(t)V(t)^{-1}$ where[2]

$$V(t) := \begin{pmatrix} X(1 + X + Y)^{-1}X(1 + Y^{-1})^{-1} & tX(1 + X + Y)^{-1}Y \\ X(1 + Y^{-1})^{-1}X(1 + X + Y)^{-1} & XY(1 + X + Y)^{-1} \end{pmatrix} \; .$$

3 Noncommutative integrability for many variables

Let $M = (M_{ij})_{1 \leq i,j \leq 3}$ be a matrix whose entries are $9 = 3 \times 3$ free independent noncommutative variables. Let us consider 3 "birational involutions"

$$\begin{aligned} I_1 &: M \mapsto M^{-1} \\ I_2 &: M \mapsto M^t \\ I_3 &: M_{ij} \mapsto (M_{ij})^{-1} \quad \forall i, j \; . \end{aligned}$$

[2] I am grateful to A.Odesskii for help in finding the matrix $V(t)$.

The composition $I_1 \circ I_2 \circ I_3$ commutes with the multiplication on the left and on the right by diagonal 3×3 matrices. We can factorize it by the action of $\mathrm{Diag}_{\mathrm{left}} \times \mathrm{Diag}_{\mathrm{right}}$ and get only 4 independent variables, setting e.g. $M_{ij} = 1$ for $i = 3$ and/or $j = 3$.

Conjecture 1. *The transformation* $(I_1 \circ I_2 \circ I_3)^3$ *is equal to the identity modulo* $\mathrm{Diag}_{\mathrm{left}} \times \mathrm{Diag}_{\mathrm{right}}$*-action. In other words, there exists two diagonal* 3×3 *matrices* $D_L(M), D_R(M)$ *whose entries are noncommutative rational functions in* 9 *variables* (M_{ij})*, such that*

$$(I_1 \circ I_2 \circ I_3)^3(M) = D_L(M) \cdot M \cdot D_R(M)\ .$$

This is a very degenerate case of integrability. Similarly, for 4×4 matrices the transformation $I_1 \circ I_2 \circ I_3$ should give a genuinely nontrivial integrable system. In the simplest case when the entries of this matrix are scalars, the Zariski closure of the generic orbit (modulo the left and the right diagonal actions) is a union of two elliptic curves.

Finally, I'll present a series of hypothetical discrete symmetries of integrable systems written as recursions. Fix an odd integer $k \geq 3$ and consider sequences $(U_n)_{n\in\mathbb{Z}}$ (of, say, $d \times d$ matrices), satisfying

$$\begin{aligned} U_n &= U_{n-k}^{-1}\,(1 + U_{n-1}\,U_{n-k+1}) &\quad \text{for } n \in 2\,\mathbb{Z} \\ U_n &= (1 + U_{n-k+1}\,U_{n-1})\,U_{n-k}^{-1} &\quad \text{for } n \in 2\,\mathbb{Z}+1\,. \end{aligned}$$

Then the map $(U_1, \dots, U_k) \mapsto (U_3, \dots, U_{k+2})$ is integrable.

4 Noncommutative Laurent phenomenon

In the previous example one observes also the noncommutative Laurent phenomenon:

$$\forall\, n \in \mathbb{Z} \qquad U_n \in \mathbb{Z}\langle U_1^{\pm1}, \dots, U_k^{\pm1}\rangle\ .$$

Also with S. Duzhin we discovered that the noncommutative birational map

$$S_l : (X, Y) \mapsto (XYX^{-1}, (1 + Y^{\,l})\,X^{-1})$$

for $l \geq 1$ satisfies the same Laurent properties, i.e. both components of 2-dimensional vector obtained by an arbitrary number of iterations, belong to the ring $\mathbb{Z}\langle X^{\pm1}, Y^{\pm1}\rangle$. The case $l = 1$ is easy, and the case $l = 2$ was studied by A. Usnich (unpublished) and by Ph. Di Francesco and R. Kedem, see [6]. The Laurent property has now three different proofs for the case $l \geq 3$ when the dynamics is *non-integrable*:

- by A. Usnich using triangulated categories, see [11],
- an elementary algebraic proof by A. Berenstein and V. Retakh, see [3],
- a new combinatorial proof of Kyungyong Lee, which also shows that all the coefficients of noncommutative Laurent polynomials obtained by iterations, belong to $\{0, 1\} \subset \mathbb{Z}$, see [8].

Finally, recently A. Berenstein and V. Retakh found a large class of noncommutative mutations related with triangulated surfaces, and proved the noncommutative Laurent property for them.

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