ALGEBRAIC FUNCTIONS OVER A FIELD OF POSITIVE CHARACTERISTIC AND HADAMARD PRODUCTS

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ABSTRACT

We give a characterization of algebraic functions over a field of positive characteristic and we then deduce that the Hadamard product of two algebraic series in several commutative variables over a field of positive characteristic is again algebraic.

1. *Introduction*

Several authors (for example, Hurwitz, Jungen, Schutzenberger, Furstenberg, Benzaghou, Fliess, Christol, etc.) have considered the Hadamard product (see Section 2 below for terminology) of two formal power series over a field, but the majority have considered this product with only one variable. We are interested in this product for several commutative variables and prove the following theorem.

The Hadamard product of two algebraic formal power series in several commutative variables over a field of positive characteristic is an algebraic formal power series.

Furstenberg [4] showed that over a finite field the Hadamard product of two algebraic series in one variable is again algebraic and this result was extended first to a perfect field of positive characteristic by Fliess [3] and then to an arbitrary field of positive characteristic by Deligne [2].

As Furstenberg [4] remarked, this result is false over a field of characteristic zero. On the other hand over any field the Hadamard product of two rational formal power series in one variable is again a rational formal power series (this is easily seen in positive characteristic and was observed by Jungen [5] in characteristic zero). The corresponding result does not hold for several variables for any field (see Remark 2 in Section 7).

We introduce a splitting process for functions (in Section 3) and define associated semilinear operators on the field of fractions of the ring of formal power series which are multiplicative with respect to the Hadamard product. We use such operators for characterizing algebraic functions as a generalization of Christol's [1] argument for one variable.

In Section 6 we prove our main theorem for a perfect field and then we remove this restriction. Finally we deduce Deligne's theorem, as an easy consequence of our main theorem.

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2. *Notation and terminology*

Let K be a field; $K[[x_1, x_2, \ldots, x_k]]$ will denote the ring of formal power series in k commuting variables $x_1, x_2, \ldots, x_k$ with coefficients in K, that is, $f \in K[[x_1, x_2, \ldots, x_k]]$ if

$$f = \sum_{\substack{n_j \geqslant 0 \\ j=1,2,\ldots,k}} a_{n_1 n_2 \ldots n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k},$$

where $a_{n_1 n_2 \ldots n_k} \in K$. We shall write $K((x_1, x_2, \ldots, x_k))$ for the field of fractions of $K[[x_1, x_2, \ldots, x_k]]$.

An element $f \in K((x_1, x_2, \ldots, x_k))$ is said to be an *algebraic function* over K if f is algebraic over the field of rational functions $K(x_1, x_2, \ldots, x_k)$. If, further, $f \in K[[x_1, x_2, \ldots, x_k]]$, then f is said to be an *algebraic series* over K. Thus in our terminology $(x_1 + x_2 + \ldots + x_k)^{\frac{1}{2}}$, $k \geqslant 1$, is not an algebraic function, because it does not lie in $K((x_1, x_2, \ldots, x_k))$. On the other hand

$$f = \sum_{n_1, n_2, n_3 \geqslant 0} \binom{n_1+n_2}{n_1} \binom{n_2+n_3}{n_2} \binom{n_3+n_1}{n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3} = [(1 - x_1 - x_2 - x_3)^2 - 4x_1 x_2 x_3]^{-\frac{1}{2}}$$

is an algebraic series with respect to any field (see [6, p. 143]).

Let ι be a non-negative vector, that is, $\iota = (n_1, n_2, \ldots, n_k)$, where $n_j \in \mathbb{N}$, $j = 1, 2, \ldots, k$. Then $\mathbf{X}^\iota$ will denote the monomial $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$. We denote by Λ the set of all non-negative vectors, and by Λ_p the set $\mathbb{Z}_p^k$, where $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$.

Throughout this paper we shall denote the ring $K[[x_1, x_2, \ldots, x_k]]$ by $K[[\mathbf{X}]]$, the field of fractions of $K[[\mathbf{X}]]$ by $K((\mathbf{X}))$, the field of rational functions $K(x_1, x_2, \ldots, x_k)$ by $K(\mathbf{X})$ and the ring of polynomials $K[x_1, x_2, \ldots, x_k]$ by $K[\mathbf{X}]$.

From now on K denotes a perfect field of characteristic $p > 0$, unless explicitly stated otherwise.

DEFINITION 2.1. Suppose that $f, g \in K[[\mathbf{X}]]$, say

$$f = \sum_{\iota \in \Lambda} a_\iota \mathbf{X}^\iota, \qquad g = \sum_{\iota \in \Lambda} b_\iota \mathbf{X}^\iota.$$

The Hadamard product of f and g, which will be denoted by $f * g$, is the series which is defined by

$$f * g = \sum_{\iota \in \Lambda} a_\iota b_\iota \mathbf{X}^\iota.$$

3. *The splitting process*

In this section we prove a fundamental lemma.

LEMMA 3.1. *If $f(\mathbf{X}) \in K[[\mathbf{X}]]$ (respectively $K((\mathbf{X}))$), then f can be written uniquely as*

$$f = \sum_{\iota \in \Lambda_p} \mathbf{X}^\iota f_\iota^p$$

for some $f_\iota \in K[[\mathbf{X}]]$ (respectively $K((\mathbf{X}))$).

Proof. Case 1, *in which $f(\mathbf{X}) \in K[[\mathbf{X}]]$.* Suppose that

$$f = \sum_{\sigma \in \Lambda} a_\sigma \mathbf{X}^\sigma.$$

Then

$$f = \sum_{\iota\in\Lambda_p,\tau\in\Lambda} a_{p\tau+\iota}\mathbf{X}^{p\tau+\iota} = \sum_{\iota\in\Lambda_p} \mathbf{X}^{\iota}\left(\sum_{\tau\in\Lambda} a_{p\tau+\iota}^{1/p}\mathbf{X}^{\tau}\right)^p.$$

Now put

$$f_\iota = \sum_{\tau\in\Lambda} a_{p\tau+\iota}^{1/p}\mathbf{X}^{\tau}. \tag{3.1.1}$$

The uniqueness part follows directly from equating coefficients.

Case 2, *in which* $f(\mathbf{X})\in K((\mathbf{X}))$. Suppose that $f=\alpha/\beta$ for some $\alpha,\beta\in K[[\mathbf{X}]]$. Then $\alpha/\beta=\alpha\beta^{p-1}/\beta^p$. Since $\alpha\beta^{p-1}\in K[[\mathbf{X}]]$, by case (1) there exist $d_\iota\in K[[\mathbf{X}]]$ such that $\alpha\beta^{p-1}=\sum_{\iota\in\Lambda_p}\mathbf{X}^\iota d_\iota^p$. Hence

$$\frac{\alpha}{\beta}=\frac{\alpha\beta^{p-1}}{\beta^p}=\sum_{\iota\in\Lambda_p}\mathbf{X}^\iota\left(\frac{d_\iota}{\beta}\right)^p,$$

where $d_\iota/\beta\in K((\mathbf{X}))$. The uniqueness part follows easily from case 1 on clearing denominators.

4. *The E operators*

In this section we define some operators on the field $K((\mathbf{X}))$ which have some very nice properties.

For $\iota\in\Lambda_p$ define

$$E_\iota : K((\mathbf{X})) \longmapsto K((\mathbf{X}))$$

by

$$E_\iota(f) = f_\iota. \tag{4.0.1}$$

Now for $f\in K((\mathbf{X}))$, by Lemma 3.1 we have (with the notation of Lemma 3.1)

$$f=\sum_{\iota\in\Lambda_p}\mathbf{X}^\iota(E_\iota(f))^p. \tag{4.0.2}$$

LEMMA 4.1. (i) E_ι *is semilinear over* K; *that is, if* $f,g\in K((\mathbf{X}))$ *and* $\lambda\in K$, *then*

(a) $E_\iota(f+g)=E_\iota(f)+E_\iota(g)$;

(b) $E_\iota(\lambda f)=\lambda^{1/p}E_\iota(f)$.

(ii) *If* $f,g\in K((\mathbf{X}))$, *then* $E_\iota(g^pf)=gE_\iota(f)$ *for each* $\iota\in\Lambda_p$.

Proof. In each case the result follows easily from (4.0.2) and Lemma 3.1.

LEMMA 4.2. *For* $\iota\in\Lambda_p$ *if* $\boldsymbol{\rho}\in\Lambda$ *with* $\boldsymbol{\rho}=p\boldsymbol{\tau}+\boldsymbol{\sigma}$, $\boldsymbol{\sigma}\in\Lambda_p$, $\boldsymbol{\tau}\in\Lambda$, *then*

$$E_\iota(\mathbf{X}^{\boldsymbol{\rho}})=\begin{cases}0 & \text{if } \boldsymbol{\sigma}\neq\iota,\\ \mathbf{X}^{\boldsymbol{\tau}} & \text{if } \boldsymbol{\sigma}=\iota.\end{cases}$$

Proof. The proof follows immediately by definition of E_ι and Lemma 4.1 (ii).

LEMMA 4.3. *If* $f,g\in K[[\mathbf{X}]]$, *then for* $\iota\in\Lambda_p$,

$$E_\iota(f*g)=E_\iota(f)*E_\iota(g).$$

Proof. The result follows directly from the definitions 2.1, (3.1.1) and (4.0.1).

5. *A characterization of algebraic functions*

In this section we generalize Christol's [1] one-variable argument from a finite field to a perfect field to show that the Hadamard product of two algebraic series in several variables over a perfect field of characteristic $p > 0$ is again algebraic.

Let Ω be the semigroup generated by the identity operator and the E_ι for $\iota \in \Lambda_p$, with ordinary composition as multiplication.

To each $f \in K((\mathbf{X}))$ we associate its orbit

$$\Omega(f) = \{E(f) : E \in \Omega\}.$$

Then we have the following.

LEMMA 5.1. *Suppose that $f \in K((\mathbf{X}))$. Then $\langle \Omega(f) \rangle$, the K-linear space spanned by $\Omega(f)$, is the smallest K-subspace of $K((\mathbf{X}))$ containing f and which is invariant under each E_ι, $\iota \in \Lambda_p$.*

Proof. Now $\langle \Omega(f) \rangle$ is a K-subspace of $K((\mathbf{X}))$ which contains f and is invariant under each E_ι by definition of $\Omega(f)$ and Lemma 4.1 (i). Every K-subspace V of $K((\mathbf{X}))$ which contains f and is invariant under each E_ι clearly also contains $\Omega(f)$ and so the result follows easily.

LEMMA 5.2. *If $f \in K((\mathbf{X}))$ is an algebraic function over K, then there exist elements $a_0, a_1, \ldots, a_N$ in $K[\mathbf{X}]$ such that*

$$\sum_{i=0}^{N} a_i f^{p^i} = 0,$$

where $a_0 \neq 0$.

Proof. Since f is algebraic over $K(\mathbf{X})$, the vector space generated by f^{p^n}, $n \in \mathbb{N}$, has finite dimension over $K(\mathbf{X})$. Hence there exist elements $a_0, a_1, \ldots, a_N$ in $K[\mathbf{X}]$ (after clearing the denominators) not all zero such that

$$\sum_{i=0}^{N} a_i f^{p^i} = 0.$$

We want to show that we may arrange that $a_0 \neq 0$. Let j be the least natural number such that there is a relation of the preceding type with $a_j \neq 0$. We shall show that $j = 0$. Suppose that $j > 0$. Since $a_j \in K[\mathbf{X}]$, by (4.0.2) we have

$$a_j = \sum_{\iota \in \Lambda_p} \mathbf{X}^\iota (E_\iota(a_j))^p. \tag{5.2.1}$$

Further, $a_j \neq 0$ and so there exists an integer vector ι such that $E_\iota(a_j) \neq 0$.

Applying E_ι to $\sum_{i=j}^{N} a_i f^{p^i} = 0$ and using Lemma 4.1 we obtain

$$\sum_{i=j}^{N} E_\iota(a_i) f^{p^{i-1}} = 0.$$

This is a relation of the preceding type where the coefficient of $f^{p^{j-1}}$ is different from zero, and this contradicts the choice of j. Hence $j = 0$ and thus we may arrange that $a_0 \neq 0$ as required.

THEOREM 5.3. *Let $f \in K((\mathbf{X}))$. Then f is an algebraic function over K if and only if there exists a finite-dimensional K-subspace V of $K((\mathbf{X}))$ such that*

(i) $f \in V$

(ii) $E_\iota(V) \subseteq V$, $\iota \in \Lambda_p$.

Proof. (Necessity) By Lemma 5.2 there exist elements $a_0, a_1, \ldots, a_N$ in $K[\mathbf{X}]$ such that

$$\sum_{i=0}^{N} a_i f^{p^i} = 0,$$

where $a_0 \neq 0$. Suppose that $g = f/a_0$. Then

$$g = \sum_{i=1}^{N} b_i g^{p^i}, \tag{5.3.1}$$

where $b_i = -a_i a_0^{p^i-2} \in K[\mathbf{X}]$.

Suppose that $\lambda = \sup(\deg a_0, \deg b_i, i = 1, 2, \ldots, N)$ where deg means the maximum degree with respect to each component of $\mathbf{X}$, and let

$$V = \left\{ h \in K((\mathbf{X})) : h = \sum_{i=0}^{N} c_i g^{p^i}, c_i \in K[\mathbf{X}], \deg c_i \leqslant \lambda \right\}.$$

Then V is a finite-dimensional K-subspace of $K((\mathbf{X}))$. Since $f = a_0 g$ and $\deg a_0 \leqslant \lambda$ it follows that $f \in V$. It remains to show that V is invariant under each E_ι for $\iota \in \Lambda_p$.

Let $h \in V$, $h = \sum_{i=0}^{N} c_i g^{p^i}$. Then

$$E_\iota(h) = E_\iota\left(c_0 g + \sum_{i=1}^{N} c_i g^{p^i}\right) = E_\iota\left(\sum_{i=1}^{N} (c_0 b_i + c_i) g^{p^i}\right) = \sum_{i=1}^{N} E_\iota(c_0 b_i + c_i) g^{p^{i-1}}$$

by (5.3.1) and Lemma 4.1. Since $\deg(c_0 b_i + c_i) \leqslant 2\lambda$, by using Lemma 4.2 $\deg E_\iota(c_0 b_i + c_i) \leqslant 2\lambda/p \leqslant \lambda$. Thus $E_\iota(h) \in V$.

(Sufficiency) Suppose that there exists a finite-dimensional K-subspace V of $K((\mathbf{X}))$ which contains f and is invariant under each E_ι. Let $n = \dim_K V$ and suppose that V_1 is the vector space generated by V over $K(\mathbf{X})$ and V_2 is the vector space generated by $\{g^p\}$, $g \in V_1$ over $K(\mathbf{X})$. Then clearly $\dim_{K(\mathbf{X})} V_1 \leqslant n$. We shall show that $V_1 = V_2$.

If $\{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ is a basis for V_1 over $K(\mathbf{X})$ then for every $g \in V_1$ we have

$$g = \sum_{i=1}^{t} c_i \alpha_i, \qquad c_i \in K(\mathbf{X}).$$

Therefore

$$g^p = \sum_{i=1}^{t} c_i^p \alpha_i^p, \qquad c_i^p \in K(\mathbf{X}),$$

which shows that $\{\alpha_1^p, \alpha_2^p, \ldots, \alpha_t^p\}$ is a system of generators of V_2. Thus

$$\dim_{K(\mathbf{X})} V_2 \leqslant \dim_{K(\mathbf{X})} V_1 \leqslant n.$$

On the other hand, for every $g \in V$ by (4.0.2) we have

$$g = \sum_{\iota \in \Lambda_p} \mathbf{X}^\iota (E_\iota(g))^p.$$

Now

$$(E_\iota(g))^p \in V^p \subseteq V_2.$$

Therefore $V \subseteq V_2$ and so $V_1 \subseteq V_2$ and thus $V_1 = V_2$.

Suppose now that B is a basis for V over K and let $L = \{\lambda : B \to \mathbb{N} \mid \lambda$ is not identically zero$\}$. Let G be the vector space over $K(\mathbf{X})$ generated by

$$\prod_{g \in B} g^{\lambda(g)} \quad \text{for } \lambda \in L.$$

As $f \in V$, we can write $f = \sum_{i=1}^{n} a_i g_i$, where $a_i \in K$ and $g_i \in B$, $i = 1, 2, \ldots, n$. Hence $f \in G$. By the binomial theorem clearly also $f^m \in G$, for $m \in \mathbb{N}$.

If $\dim_{K(\mathbf{X})} G < \infty$, then $f, f^2, f^3, \ldots, f^N$ will be linearly dependent over $K(\mathbf{X})$ for a suitable positive integer N. Hence there exist elements $c_1, c_2, \ldots, c_N$ in $K(\mathbf{X})$ not all zero such that $\sum_{i=1}^{N} c_i f^i = 0$. Thus f is algebraic over $K(\mathbf{X})$. Hence it is enough to show that the dimension of G over $K(\mathbf{X})$ is finite.

Suppose that $g \in B \subseteq V \subseteq V_1$. So $g^p \in V_2$. As $V_1 = V_2$, $g^p \in V_1$ is a linear combination of the elements of the basis B (of V over K) with coefficients in $K(\mathbf{X})$. Continuing this process it follows easily that G is generated by the $\prod_{g \in B} g^{\lambda(g)}$ (where $\lambda \in L$ with each $\lambda(g) < p$) over $K(\mathbf{X})$. Hence $\dim_{K(\mathbf{X})} G \leqslant p^n - 1$.

COROLLARY 5.4. *Suppose that $f \in K((\mathbf{X}))$. Then f is an algebraic function over K if and only if* $\dim_K \langle \Omega(f) \rangle$ *is finite.*

Proof. (Necessity) If f is an algebraic function over K, then by Theorem 5.3 there exists a finite-dimensional K-subspace V of $K((\mathbf{X}))$ such that $f \in V$ and V is invariant under each E_ι for $\iota \in \Lambda_p$. By Lemma 5.1 $\langle \Omega(f) \rangle \subseteq V$ and hence $\dim_K \langle \Omega(f) \rangle$ is finite.

(Sufficiency) By Theorem 5.3 it is enough to take $V = \langle \Omega(f) \rangle$.

REMARK 1. Suppose that $f \in K[[\mathbf{X}]]$; then $\langle \Omega(f) \rangle \subseteq K[[\mathbf{X}]]$. Hence if f is an algebraic function over K, then by Corollary 5.4 $\langle \Omega(f) \rangle$ is a finite-dimensional K-subspace of $K[[\mathbf{X}]]$ (and not just of $K((\mathbf{X}))$) which contains f and is invariant under each E_ι for $\iota \in \Lambda_p$.

COROLLARY 5.5. *Suppose that $f, g \in K[[\mathbf{X}]]$. If f, g are algebraic series over K, then $f * g$ is again an algebraic series over K.*

Proof. Since f and g are algebraic series over K, by Corollary 5.4 and Remark 1 there exist finite-dimensional K-subspaces V_f and V_g of $K[[\mathbf{X}]]$, such that $f \in V_f$, $g \in V_g$ and V_f, V_g are invariant under each E_ι for $\iota \in \Lambda_p$. Suppose that $V_f = \langle \alpha_t : 1 \leqslant t \leqslant n \rangle$ and $V_g = \langle \beta_s : 1 \leqslant s \leqslant m \rangle$. Define $V_f * V_g = \langle \alpha_t * \beta_s : 1 \leqslant t \leqslant n, 1 \leqslant s \leqslant m \rangle$. Then $V_f * V_g$ is a finite-dimensional K-subspace of $K[[\mathbf{X}]]$ which we shall show satisfies the required conditions; that is,

(i) $f * g \in V_f * V_g$,
(ii) $E_\iota(V_f * V_g) \subseteq V_f * V_g$ for $\iota \in \Lambda_p$.

From the K-bilinearity of the Hadamard product $*$, we get (i). To establish (ii) suppose that $h \in V_f * V_g$ and so

$$h = \sum_{s=1}^{m} \sum_{t=1}^{n} \lambda_{ts} (\alpha_t * \beta_s),$$

where $\lambda_{ts} \in K$. For each $\iota \in \Lambda_p$

$$E_\iota(h) = \sum_{s=1}^{m} \sum_{t=1}^{n} \lambda_{ts}^{1/p} E_\iota(\alpha_t * \beta_s) = \sum_{s=1}^{m} \sum_{t=1}^{n} \lambda_{ts}^{1/p} E_\iota(\alpha_t) * E_\iota(\beta_s)$$

by Lemmas 4.1 and 4.3.

Since $E_\iota(\alpha_t) \in V_f$ and $E_\iota(\beta_s) \in V_g$ for $1 \leqslant t \leqslant n$, $1 \leqslant s \leqslant m$, it follows from part (i) that

$$E_\iota(\alpha_t) * E_\iota(\beta_s) \in V_f * V_g$$

and hence

$$E_\iota(h) \in V_f * V_g.$$

Thus $f * g$ is an algebraic series over K (by Theorem 5.3).

6. *The proof of the main theorem*

In the previous section we showed that over a perfect field of characteristic $p > 0$, the Hadamard product of two algebraic series in k variables is again an algebraic series; that is, we have proved the main theorem with the additional assumption that K is perfect. The next result enables us to remove this restriction.

THEOREM 6.1. *Suppose that K is any field. If $h \in K((\mathbf{X}))$ is an algebraic function over L, where L is an extension field of K, then h is an algebraic function over K.*

Proof. Since h is algebraic over $L(\mathbf{X})$ there exist a_i, $i = 0, 1, 2, \ldots, n$, elements of $L[\mathbf{X}]$ (after clearing the denominators) not all zero such that

$$\sum_{i=0}^{n} a_i(\mathbf{X})\, h^i(\mathbf{X}) = 0, \tag{6.1.1}$$

where n is the degree of h over $L(\mathbf{X})$. For each $j = 0, 1, 2, \ldots, n$, $a_j = \sum_\iota b_{j\iota} \mathbf{X}^\iota$ (finite sum) and from above there exists some coefficient $b_{j\rho} \in L$ which is non-zero.

Let $b_{j\rho}$ be the first element of a basis B for L over K. Define a K-linear map $\phi\colon L \to K$ such that if $x \in B$ then

$$\phi(x) = \begin{cases} 1 & \text{if } x = b_{j\rho}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if we denote $\phi(x)$ by $\tilde{x}$, then from (6.1.1) we get

$$\sum_{i=0}^{n} \tilde{a}_i(\mathbf{X})\, h^i(\mathbf{X}) = 0,$$

where the finite sum

$$\tilde{a}_t(\mathbf{X}) = \sum_\iota \tilde{b}_{t\iota} \mathbf{X}^\iota$$

is a non-zero element of $K[\mathbf{X}]$ for some t, by the choice of ϕ. Thus h is an algebraic function over K.

We are now in a position to prove the main theorem.

THEOREM. *If K is a field of characteristic $p > 0$ and if f, g are algebraic series over K, then $f * g$ is again an algebraic series over K.*

Proof. Suppose that L is a perfect extension of K, for example, the algebraic closure of K. Then f and g are algebraic series over L. Hence by Corollary 5.5, $f*g$ is an algebraic series over L and so by Theorem 6.1 $f*g$ is an algebraic series over K.

7. *A theorem of Deligne*

Deligne's theorem [**2**] can be proved directly from the main theorem.

THEOREM 7.1. *Suppose that K is a field of characteristic $p > 0$. If $f \in K[[\mathbf{X}]]$ and $f = \sum_{\sigma\in\Lambda} a_\sigma \mathbf{X}^\sigma$ is an algebraic series in $\mathbf{X}$ over K, then $I(f) = \sum_{n\geqslant 0} a_{n.1} t^n$ is an algebraic series in t over K.*

Proof. Since

$$g = \sum_{n\geqslant 0} (x_1 x_2 \dots x_k)^n = \frac{1}{1 - x_1 x_2 \dots x_k}$$

is an algebraic (in fact rational) series over K it follows that

$$h = f*g = \sum_{n\geqslant 0} a_{nn\dots n} (x_1 x_2 \dots x_k)^n$$

is an algebraic series in $\mathbf{X}$ over K by the main theorem.

Let $t = x_1 x_2 \dots x_k$. We want to show that h is an algebraic series in t over K. Suppose that N is the degree of h over $K(\mathbf{X})$. Since

$$K(\mathbf{X}) = K(x_1 x_2 \dots x_k, x_2, \dots, x_k) = K(t, x_2, \dots, x_k),$$

h is algebraic over $K(t, x_2, x_3, \dots, x_k)$, and so it follows that for $0 \leqslant i \leqslant N$ there exist $b_i(t, x_2, x_3, \dots, x_k)$ in $K[t, x_2, x_3, \dots, x_k]$ (after clearing the denominators), not all zero, such that

$$\sum_{i=0}^{N} b_i(t, x_2, x_3, \dots, x_k)\, h^i = 0. \tag{7.1.1}$$

Since the left-hand side of (7.1.1) can be regarded as a polynomial in $x_2, x_3, \dots, x_k$ with coefficients in $K[[t]]$ (in fact, with coefficients of the form of polynomial expressions in h with coefficients in $K[t]$), and since not all the b_i are zero, equating the coefficients of the various monomials in $x_2, x_3, \dots, x_k$ to zero yields at least one non-trivial equation for h of the desired form. Hence h is an algebraic series in t over K.

Deligne [**2**] raises some questions concerning the estimation of the degree of $I(f)$ over $K(t)$ at the end of his paper. His estimates are based on his deep geometric interpretation of the main theorem.

It would be interesting to obtain similar estimates for the degrees of the Hadamard products which are considered in this paper but it is not clear how to obtain such sharp estimates using the algebraic methods developed here.

REMARK 2. It follows from the main theorem that *the Hadamard product of two rational series in k variables over a field of positive characteristic is algebraic.*

Jungen [**5**] showed that over a field of characteristic zero the Hadamard product of two rational series in one variable is again a rational series. This result is also true in a field of positive characteristic. However, this is not true in more than one variable over any field as the following example shows. If

$$f = \sum_{n,m \geqslant 0} \binom{n+m}{n} x^n y^m = \frac{1}{1-x-y},$$

which is rational, then

$$f * f = \sum_{n,m \geqslant 0} \binom{n+m}{n}^2 x^n y^m = \{(1-x-y)^2 - 4xy\}^{-\frac{1}{2}},$$

and this is not rational.

Note. Recently Lipshitz has informed us that he and Denef have obtained related results to those in this paper. (These have now appeared in the *Journal of Number Theory* 26 (1987) 46–67.)

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