Transcendence via Differential Galois theory

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The goal is to explain how the Galois theory of differential or difference equations can be used to show that certain power series/special functions/generating series are transcendent or differentially transcendent (also called hypertranscendent).

A nice example is Hölder's theorem: *The Gamma function $\Gamma(x)$ is differentially transcendent over $\mathbb{C}(x)$, i.e., the functions $\Gamma(x), \Gamma'(x), \Gamma''(x), \ldots$ are algebraically independent over $\mathbb{C}(x)$.*

Hölder's theorem can be proved by using a certain Galois theory that applies to linear difference equations. Note that $\Gamma(x)$ is a solution of the linear difference equations $f(x+1) = xf(x)$. This method generalizes to yield a criterion to decide whether a solution to the linear difference equation $f(x+1) = h(x)f(x)$ (where $h(x) \in \mathbb{C}(x)$) is differentially transcendent. (Details can be found in [HS08].)

A key idea of the proof is that the algebraic relations among the functions $\Gamma(x), \Gamma'(x), \Gamma''(x), \ldots$ correspond to equations defining the Galois group of $f(x+1) = xf(x)$. Here the Galois group is a subgroup of $\mathbb{G}_m = \mathrm{GL}_1$ defined by algebraic differential equations (a so-called linear differential algebraic group). These groups are classified and well-understood. They are all defined by an equation of a very particular form. Such an equation then gives rise to a very particular algebraic relation among $\Gamma(x), \Gamma'(x), \Gamma''(x), \ldots$. So the Galois theory tells us that if there is an algebraic relation among $\Gamma(x), \Gamma'(x), \Gamma''(x), \ldots$, then there has to exist a very particular algebraic relation. This particular algebraic relation can then be ruled out easily.

This strategy of proof has been employed by several authors in various setups. Here are some examples:

- Generating series obtained from certain automatic sequences are differentially transcendent ([DHR]). The generating series satisfy Mahler-difference equations.
- The incomplete Gamma function $\gamma(t,x) = \int_0^x s^{t-1}e^{-s}ds$ which satisfies a second order linear $\frac{\partial}{\partial x}$-differential equation is $\frac{\partial}{\partial t}$-differentially transcendent ([Arr13]).
- Generating series of walks in the quarter plane (for certain admissible step sets) are differentially transcendental ([DHSR]).

A standard reference for the Galois theory of linear differential equations is [vdPS03] and for linear difference equations [vdPS97]. A lighter introduction to the Galois theory of linear differential equations would be [Sin09]. For the applications to differential transcendence one needs the so-called parameterized Galois theories ([CS07], [HS08]) where one has two (or more) operations, e.g., $f(x) \mapsto f(x+1)$ and $f(x) \mapsto f'(x)$ for the above example with the Gamma function. Here are two introductory papers to the topic of using Galois theories to prove differential transcendence: [DV12], [Har16]. The two talks with title "Algorithmic aspects of Galois theories for functional equations and hypertranscendence" given by Carlos Arreche this summer at the Fields Institute might also be a good point to start. You can find videos of the talks here: `http://www.fields.utoronto.ca/activities/17-18/differential-galois`.

References

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