Basic Material for Lisbon 2020

Sergey Yurkevich and Paul Zellhofer

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1 Regular Singularities and Monodromy Groups

1.1 Motivation

We want to study ordinary linear differential equations

$$y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_0(x)y(x) = 0, \tag{1}$$

where $a_i(x)$ are meromorphic functions.[1] Because of a theorem of Cauchy, around a (for now finite) point $a \in \mathbb{C}$ which is not a pole of any $a_i(x)$, there always exist n linearly independent solutions. A prominent method for solving such equations at *any* point $a \in \mathbb{C}$ is the so-called *method of Frobenius*. Vaguely speaking, around $x_0 = a$, it starts with an ansatz of the form

$$y(x) = (x-a)^\alpha \sum_{k=0}^{\infty} A_k (x-a)^k,$$

then formal differentiation and substitution back into (1). By comparison of coefficients of powers of x, one may hope to find a recursion for the A_i's. In general, the Frobenius method may be applied to find possible solutions that are power series times complex powers $(x-a)^\alpha$ near some given $a \in \mathbb{P}^1(\mathbb{C})$, however a priori it is not at all clear that such a solution will exist. Nevertheless, we still hope for n linearly independent solutions this way.

It can be shown that this method works for so-called ordinary points a, i.e. those points $a \in \mathbb{C}$ for which all $a_i(a)$ are defined. Moreover even if a is singular (i.e. not ordinary) but suffices a certain condition (see below), the Frobenius method can still be made to work and again provide n independent solutions near a. The goal of this section is to provide a framework of proper definitions to deal with these kinds of problems.

The general references for this topic are [Sin77] and [Poo60]. Also the works of [Waa02] and [Li02] give a good introduction and we will cite them at some points.

[1] For simplicity we will often assume that all $a_i(x) \in \mathbb{C}(x)$ are even rational.

1.2 Definitions

Consider a linear differential operator

$$L = \frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x), \tag{2}$$

where each $a_i(x) \in \mathbb{C}(x)$ is a rational function in the variable x. Let $a \in \mathbb{P}^1(\mathbb{C})$ be a point of the Riemann Sphere which we shall identify with $\mathbb{C} \cup \{\infty\}$. A point $a \neq \infty$ is called *regular point of* L if $(x-a)^i a_{n-i}(x)$ has no pole at $x = a$ for all $i = 1, \ldots, n$. Infinity is a regular point if $\lim_{x\to\infty} x^i a_{n-i}(x)$ exists for all $i = 1, \ldots, n$. In other words, a (finite) point a is a regular point of (2) if each $a_i(x)$ has a pole at a of order at most $n - i$. This algebraic definition of regular points of differential operators is due to Lazarus Immanuel Fuchs (*1833; †1902) and is known under the name *criterion of Fuchs*. A point which is not regular is called *irregular point*. In [Sin77] M. Singer defines the same notion by an equivalent analytic statement: $a \in \mathbb{P}^1(\mathbb{C})$ is called regular point of L if for any holomorphic solution y of $Ly = 0$ for some angular sector $\mathcal{S}$, there exists a natural number N such that

$$\begin{cases} \lim_{x\to a}(x-a)^N y(x) = 0 & \text{if } a \neq \infty, \\ \lim_{x\to\infty} x^{-N} y(x) = 0 & \text{if } a = \infty. \end{cases}$$

A finite point $a \in \mathbb{P}^1(\mathbb{C})$ is called a *singular point of* (2) if, for some i, $a_i(x)$ has a pole at $x = a$. Infinity is called a singular point if, after one makes the change of variable $t = 1/x$, the new operator has $t = 0$ as a singular point. This is equivalent to the condition that $\lim_{x\to\infty} x^{2i} a_{n-i}(x)$ does not exist for some i, because $d/dx = -t^2 d/dt$. A point which is not singular is said to be an *ordinary point of* (2). By the criterion of Fuchs, we see immediately that all ordinary points are regular. It is important to bear in mind that other than that the notions *regular/irregular* and *singular/ordinary* can exist together; consider for example the following table with the three possible scenarios for the point $a = 0$:

	regular	irregular
ordinary	$(x+1)y'' + xy' + y = 0$	-
singular	$xy'' + (2-x)y' - y = 0$	$x^2y'' + (2-x)y' - y = 0$

Finally, we say that a linear differential operator L if of *Fuchsian type* if all singular points of L are regular. A prominent example of a non-Fuchsian operator is the Bessel equation

$$x^2y'' + xy' + (x^2 - k^2)y = 0.$$

Here, it is evident that 0 is singular and regular not causing problems, however the point ∞ is an irregular singularity.

According to [Sin77], the following proposition follows directly from the analytic definition of regular points and a well-known fact about algebraic functions:

Proposition 1 *Let L be a linear differential operator like in in* (2) *and assume that $Ly = 0$ has n linearly independent solutions, each algebraic over $\mathbb{C}(x)$. Then L is of Fuchsian type.*

Finally, we state another proposition from [Sin77], trying to motivate the usefulness of the definitions above.

Proposition 2 *Let L be a linear differential operator of Fuchsian type with singular points $\{a_1, \dots, a_m\}$. Assume $y(x)$ is a solution of $Ly = 0$ and $y'/y \in \mathbb{C}(x)$ is a rational function. Then y is of the form*

$$y(x) = g(x) \prod_{i=1}^{m} (x - a_i)^{e_i},$$

for some polynomial $g(x)$.

1.3 The Theorem of Fuchs

In order to explain the theorem to follow we start with an example. Consider the linear ordinary differential equation of second order

$$y'' + p(x)y' + q(x)y = 0,$$

with meromorphic functions $p(x), q(x)$. We assume that 0 is a regular singular point, hence $p(x)$ has a pole at 0 of order (at most) 1 and $q(x)$ has a pole at 0 or order (at most) 2. We can write

$$p(x) = \frac{1}{x} \sum_{n=0}^{\infty} p_n x^n, \qquad q(x) = \frac{1}{x^2} \sum_{n=0}^{\infty} q_n x^n.$$

As already explained in the beginning, we want to find two solutions of the form

$$y_j(x) = x^{\alpha_j} \sum_{n \geq 0} y_{j,n} x^n, \qquad j \in \{1, 2\}.$$

The comparison of the coefficient of $x^{\alpha-2}$ yields the *indicial equation* for α given by $\alpha^2 + (p_0 - 1)\alpha + q_0 = 0$. Denote the solutions of this quadratic equation by α_1 and α_2 and assume that $\Re(\alpha_1) \geq \Re(\alpha_2)$. Then, by the *theorem of Fuchs*, there are two cases:

1. If $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then there exist two solutions of the form

$$y_1(x) = x^{\alpha_1} \sum_{n \geq 0} y_{1,n} x^n \text{ and}$$
$$y_2(x) = x^{\alpha_2} \sum_{n \geq 0} y_{2,n} x^n.$$

2. If $\alpha_1 - \alpha_2 \in \mathbb{Z}$, then there exist two solutions of the form

$$y_1(x) = x^{\alpha_1} \sum_{n \geq 0} y_{1,n} x^n \text{ and}$$
$$y_2(x) = x^{\alpha_2} \sum_{n \geq 0} y_{2,n} x^n + c \ln(x) y_1(x).$$

To make the example even more explicit, consider Kummer's equation

$$xy'' + (b - x)y' - ay = 0, \tag{3}$$

with $a = 1, b = 2$ and around $x = 0$ (clearly, by the criterion of Fuchs this is a regular singular point). After dividing by x, we have $p(x) = (2 - x)/x$ and $q(x) = -1/x$, hence $p_0 = 2$ and $q_0 = 0$. This results in the characteristic equation $\alpha^2 + \alpha = 0$, which has solutions $\alpha_1 = 0$ and $\alpha_2 = -1$, so we are in the second case. Now, the two independent solutions of the well-known equation (3) in the case $a = 1$ and $b = 2$ are given by $1/x$ and e^x/x. On the first glance this does not align with the theorem of Fuchs above, however the theory indeed works if we consider

$$y_1(x) = (e^x - 1)/x \text{ and}$$
$$y_2(x) = 1/x.$$

Of course, the theorem of Fuchs works for orders greater 2 as well. To state it, we define indicial equation properly:
Let L be a linear differential operator like in (2) and $a \in \mathbb{P}^1(\mathbb{C})$ a regular singular point. If a is finite, set $a_i = \lim_{x \to a} (x - a)^{n-i} a_i(x)$ and $a_i = \lim_{x \to \infty} x^{n-i} a_i(x)$ in the other case. Then the *indicial equation* for finite a is given by

$$x(x-1)\cdots(x-n+1) + a_{n-1} x(x-1)\cdots(x-n+2) + \cdots + a_{n-1}x + a_1 = 0,$$

and for ∞ the *indicial equation* is defined by

$$x(x+1)\cdots(x+n-1) - a_{n-1} x(x+1)\cdots(x+n-2) + \cdots + (-1)^n a_1 = 0.$$

A solution of of the indicial equation at some $a \in \mathbb{P}^1(\mathbb{C})$ is called *local exponent at a*. Note that if a is finite and ordinary then the local exponents at a are precisely $0, 1, \ldots, n-1$.

Theorem 1 (Fuchs) *Let L be a linear differential operator and $a \in \mathbb{P}^1(\mathbb{C})$ a regular singularity. Define $t = 1/x$ in the case of $a = \infty$ and $t = x - a$ otherwise. Suppose that α is a local exponent at a such that none of the numbers $\alpha+1, \alpha+2, \ldots$ is also a local exponent. Then there exists a holomorphic power series $g(t)$ in t with non-zero constant term such that $y = t^\alpha g(t)$ is a solution of $Ly = 0$.*

A simple residue counting argument yields the following result, also attributed to Fuchs:

Theorem 2 (Fuchs relation) *Let $\alpha_1(a), \alpha_2(a), \dots, \alpha_n(a)$ denote the local exponents of a Fuchsian equation of order n at any $a \in \mathbb{P}^1(\mathbb{C})$. Then it holds*

$$\sum_{a \in \mathbb{P}^1(\mathbb{C})} \left(\alpha_1(a) + \alpha_2(a) + \cdots + \alpha_n(a) - \binom{n}{2} \right) = -2\binom{n}{2}.$$

Note that the sum above is finite since for finite ordinary points we already observed that the local exponents are $0, 1, \dots, n-1$ and hence their sum conveniently cancels with the binomial coefficient.

1.4 Monodromy Groups

We start with two examples in order to illustrate the connection of our framework with group theory.

Consider the second order linear differential equation

$$x^2 y'' + \frac{1}{6} x y' + \frac{1}{6} y = 0, \tag{4}$$

around $x = 0$, i.e. the famous Euler equation with parameters $a = b = 1/6$. As any Euler equation, the ODE above is of Fuchsian type and we may apply the same procedure as in the example above. We obtain the indicial equation $\alpha^2 - 5\alpha/6 + 1/6$ and the solutions $\alpha_1 = 1/2, \alpha_2 = 1/3$. One easily sees that

$$y_1(x) = x^{1/2} \quad \text{and} \quad y_2(x) = x^{1/3}$$

are the two linearly solutions of (4) at $x = 0$. We can continue $x^{1/2}$ and $x^{1/3}$ analytically in the complex plane $\mathbb{C}$ around 0. If we do this along any positively oriented single closed path, then the basis of solutions becomes $(-x^{1/2}, e^{2\pi i/3} x^{1/3})$. The effects of the path on the basis of solutions can be represented by left multiplication with the 2×2 matrix

$$\gamma = \begin{pmatrix} -1 & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}.$$

Instead of focusing on one specific path, one might also consider all closed paths around 0 in $\mathbb{C}$. Each path gives rise to a matrix representing the change of basis. Such a matrix equals γ^n, where n is the winding number of the path around 0. All matrices that are obtained in this way form a group, which is generated by γ. This group is known as the *monodromy group* of equation (4).

Now consider the Euler differential equation with parameters $(-1, 1)$:

$$x^2 y'' - x y' + y = 0, \tag{5}$$

again near $x = 0$. The same procedure as before yields $\alpha_1 = \alpha_2 = 1$, so we are in the second case of the theorem of Fuchs. Indeed the solutions we find are

$$y_1(x) = x \quad \text{and} \quad y_2(x) = x \ln(x).$$

As before, we can consider the analytical continuation of these solutions along any closed path in $\mathbb{C} \setminus \{0\}$. Such a path changes $x\ln(x)$ into $2\pi ikx + x\ln(x)$ for a certain integer k. It transforms the above basis of solutions into $(x, 2\pi ikx + x\ln(x))$. This transformation can again be viewed as a 2×2 matrix multiplication with an element of the group

$$\left\{ \begin{pmatrix} 1 & 2\pi ik \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\},$$

which is the monodromy group of equation (5).

Notice that the differential equation (4) has a *finite* monodromy group of order 6 and all solutions are *algebraic functions*. On the other hand, the monodromy group of the equation (5) is *infinite* and we also have a *transcendental* solution $x\ln(x)$. This is a consequence a theorem to follow, but First we will define the monodromy group properly.

Let L be the linear differential operator as in (2) and $S \subseteq \mathbb{P}^1(\mathbb{C})$ its set of singular points. Moreover choose a fixed point $x_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$. There exists a basis $f_1, f_2, \ldots, f_n$ of holomorphic solutions at x_0. These functions can be continued analytically along any closed path u in $\mathbb{P}^1(\mathbb{C}) \setminus S$ that begins and ends in x_0. The functions $f_i, i = 1, 2, \ldots, n$ then change into new functions $\widetilde{f}_i$, $i = 1, 2, \ldots, n$. The newly obtained functions are still independent solutions of the differential equation. Therefore, they must be linear combinations of the original basis $\underline{f} = (f_1, \ldots, f_n)$. This implies that there exists an invertible matrix $M_{\underline{f}}(u) \in \mathrm{GL}(n, \mathbb{C})$ with the property

$$\begin{pmatrix} \widetilde{f}_1 \\ \widetilde{f}_2 \\ \vdots \\ \widetilde{f}_n \end{pmatrix} = M_{\underline{f}}(u) \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Note that $M_{\underline{f}}(u)$ depends on the given differential equation (2), since $\underline{f}$ does. For convenience one usually writes $M(u)$ instead of $M_{\underline{f}}(u)$ if it is clear that the involving matrix is taken with respect to the ordered basis $\underline{f}$.

If u is a closed path with none of the singular points enclosed, then $M(u)$ is obviously the identity-matrix. The matrix $M(u)$ is in fact determined by the class $[u]$ of u in the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, x_0)$ of $\mathbb{P}^1(\mathbb{C}) \setminus S$ with base point x_0. In this way one obtains the so-called *monodromy representation*

$$\begin{aligned} M_{\underline{f}} : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, x_0) &\to \mathrm{GL}(n, \mathbb{C}) \\ [u] &\mapsto M_{\underline{f}}(u). \end{aligned}$$

Clearly it depends chosen basis $\underline{f}$ and on the base point x_0. A simple argument shows that the choice of another basis $\underline{g}$ results in just a conjugation of the matrices $M_{\underline{f}}(u)$. The conjugacy class of $M_{\underline{f}}(\pi_1(\mathbb{P}^1(\mathbb{C}))$ is called the *monodromy group of* L and is denoted by $M_{\underline{f}}$ by a slight abuse of notation.

Any solution f of $L(y) = 0$ is a $\mathbb{C}$-linear combination $\sum_{i=1}^n a_i f_i$ of the basis vectors of $\underline{f}$. The analytic continuation of f after completing a closed path u as before becomes $\sum_{i=1}^n a_i \widetilde{f_i}$. This is exactly $\sum_{i=1}^n a_i (M_{\underline{f}}(u)\underline{f}^T)_i$, in which the summation is taken over the n entries of $M_{\underline{f}}(u)\underline{f}^T$. This description of the analytic continuation of f gives rise to the natural action of $M_{\underline{f}}$ on the solution space of $L(y) = 0$. For a monodromy matrix $\gamma \in M_{\underline{f}}$ we will write γf for this action on a function f satisfying $L(f) = 0$.

We get the following simple fact if we consider simple loops in $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, x_0)$, each with exactly one singular point inside.

Proposition 3 *The monodromy group of an ordinary linear differential equation of order n with singular points $S \subseteq \mathbb{P}^1(\mathbb{C})$ is generated by $|S|$ matrices $\gamma_1, \gamma_2, \ldots, \gamma_{|S|}$ that satisfy $\gamma_1\gamma_2 \cdots \gamma_{|S|} = I_n$.*

The last theorem of this section is a famous description about when a linear ODE has only algebraic solutions.

Theorem 3 *An equation of Fuchsian type has only algebraic solutions if and only if its monodromy group is finite.*

We shall give a rough idea of the proof and refer to [Waa02, pp. 24] and [Sin77, pp. 9] for the full story. To show ($\Rightarrow$), one chooses a basis $\{f_1, \ldots, f_n\}$ of algebraic solutions for $Ly = 0$. Then the key observation is that under analytic continuation along any closed loop on $\mathbb{P}^1(\mathbb{C})$ any algebraic solution changes to one of its conjugates. However an algebraic function has only a finite number of conjugates and therefore the number of images of $\{f_1, f_2, \ldots, f_n\}$ under monodromy is also finite. It follows that the monodromy group of L must be finite as well. For the other direction, one constructs given a solution f of $Ly = 0$ the polynomial $P(t) = \prod_{\gamma \in M}(t - \gamma f)$, where M denotes the finite monodromy group. Since $P(f) = 0$, it suffices to show that each coefficient of $P(t)$ is a rational function. However this follows from the fact that $P(t)$ is invariant under the action of M and a simple lemma.

We conclude this section by defining *Picard-Fuchs differential operators*. In the literature there seem to be many equivalent formulations. In his dissertation [Li02] D. Li defines a differential operator of order n to be of Picard-Fuchs type if it is Fuchsian and its monodromy group is contained in $\mathrm{SL}(n, \overline{\mathbb{Q}})$. Kontsevich and Zagier add in [KZ01] the following two definitions:

- Picard-Fuchs equations are those for which the power series expansion of every solution at a chosen (rational) base point t_0 has coefficients whose numerators and denominators grow at most exponentially.

- An equation is called Picard-Fuchs if its differential operator has nilpotent p-curvature for almost every prime p.

In [Kat72] Nicholas M. Katz writes "Recall that if $K/\mathbb{C}$ is any function field, and U/K any smooth K-variety, the finite-dimension K-spaces of algebraic de

Rham cohomology $H^n_{DR}(U/K)$ are each endowed with a canonical integrable connection ∇, that of Gauss-Manin ("differentiation ofcohomology classes with respect to parameters"). The resulting differential equations $(H^n_{DR}(U/K), \nabla)$ are called the Picard-Fuchs equations."

2 Hypergeometric Functions and Generalizations

2.1 Some Historic Information

The term "hypergeometric series" was first used by John Wallis in 1655. Hypergeometric series were studied by Leonhard Euler, but the first full systematic treatment was given by Carl Friedrich Gauss (1813). Studies in the nineteenth century included those of Ernst Kummer (1836), and the fundamental characterisation by Bernhard Riemann (1857) of the hypergeometric function by means of the differential equation it satisfies. Riemann showed that every second-order linear ODE with three regular singular points can be transformed into an equation satisfied by a hypergeometric function. The cases where the solutions are algebraic functions were found by Hermann Schwarz (Schwarz's list, see below).

2.2 The Gaussian Hypergeometric Function

The Gaussian or ordinary hypergeometric function is formally defined for complex parameters a, b, c with $c \notin \mathbb{Z}_{\leq 0}$ as the following power series in x:

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, $$

where $(q)_n$ is the Pochhammer symbol, defined by

$$ (q)_n = \begin{cases} 1 & n = 0, \\ q(q+1)\cdots(q+n-1) & n > 0. \end{cases} $$

Many of the common mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Some typical examples are

$$ \begin{aligned} \ln(1-x) &= -x \cdot {}_2F_1(1, 1; 2; x), \\ \frac{1}{(1-x)^a} &= {}_2F_1(a, 1; 1; x), \\ \arcsin(x) &= x \cdot {}_2F_1\big(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\big). \end{aligned} $$

The hypergeometric function ${}_2F_1(a, b; c; x)$ is a solution of Euler's hypergeometric differential equation

$$ x(1-x)\, y'' + [c - (a+b+1)x]\; y' - ab\, y = 0. \tag{6} $$

Following the definitions from the previous section, we see by the criterion of Fuchs that (6) has exactly three regular singular points, namely at $0, 1$ and ∞. Even though this fact is purely computational, we shall prove it here, since it may make the notion of regular singular points more familiar to the reader.

We rewrite (6) in monic form:

$$y'' + p(x)y' - q(x)y = 0,$$

with

$$p(x) = \frac{c - (a+b+1)x}{x(1-x)} = \frac{c}{x} + \frac{c-a-b-1}{1-x} \quad \text{and}$$
$$q(x) = \frac{ab}{x(1-x)}.$$

It follows by definition that 0 and 1 are regular singular. In order to check ∞ we make the transformation $t = 1/x$ and with $dt/dx = -1/x^2 = -t^2$ and some short computation we arrive at the new differential equation

$$y''(t) + \tilde{p}(t)y'(t) - \tilde{q}(t)y(t) = 0,$$

with

$$\tilde{p}(t) = \frac{c+2}{t} + \frac{c-a-b-1}{t(t-1)} \qquad \text{and} \qquad \tilde{q}(t) = \frac{ab}{t^2(t-1)}.$$

Indeed 0 is a singular point of the new equation, and therefore so is ∞ of the original one. It is also clearly regular since

$$\lim_{x\to\infty} xp(x) = a+b+1 \text{ and } \lim_{x\to\infty} x^2 q(x) = -ab.$$

The local exponents at these regular singularities are:

- at 0 given by $\alpha_1 = 0, \alpha_2 = 1 - c$
- at 1 given by $\alpha_1 = 0, \alpha_2 = c - a - b$
- at ∞ given by $\alpha_1 = a, \alpha_2 = b$

As one might guess, all solutions to the hypergeometric differential equation 6 are built out of the hypergeometric series ${}_2F_1(a, b; c; x)$. At each of the three singular points $0, 1, \infty$, there are two special solutions of the form x^α times a holomorphic function of x, where α is one of the two roots of the indicial equation given above. This gives $3 \cdot 2 = 6$ special solutions, as follows.

1. Around the point $x = 0$. Assume c is not an integer, then the two solutions are simply

$$y_1(x) = {}_2F_1(a, b; c; x) \qquad \text{and}$$
$$y_2(x) = x^{1-c}\, {}_2F_1(1 + a - c, 1 + b - c; 2 - c; x).$$

If $c \in \mathbb{Z}$ the situation becomes more difficult and shall not be covered here.

2. Around the point $x = 1$. We have to assume that $c - a - b \notin \mathbb{Z}$, then the two solutions are given by

$$y_1(x) = {}_2F_1(a, b; 1 + a + b - c; 1 - x),$$
$$y_2(x) = (1 - x)^{c-a-b}\, {}_2F_1(c - a, c - b; 1 + c - a - b; 1 - x).$$

   Again, if the condition on $c - a - b$ is not satisfied, there exist other solutions that are more complicated.

3. Around $x = \infty$ and for for $a - b$ not integer we find

$$y_1(x) = x^{-a}\, {}_2F_1\left(a, 1 + a - c; 1 + a - b; x^{-1}\right),$$
$$y_2(x) = x^{-b}\, {}_2F_1\left(b, 1 + b - c; 1 + b - a; x^{-1}\right),$$

   with the more complicated but resolved case $a - b \in \mathbb{Z}$.

2.3 A Generalization: Riemann's Differential Equation

We claimed before that every second-order linear ODE with three regular singular points can be transformed into an equation satisfied by a hypergeometric function. Now we shall justify this claim.

Let a linear ordinary differential equation of second order with three regular singular points at a, b, c be given. We know that near each singularity $s \in \{a, b, c\}$ the equation admits two linearly independent solutions of the form $x^{\alpha_1(s)} f(x)$ and $x^{\alpha_2(s)} g(x)$. Moreover, the relation of Fuchs yields

$$\alpha_1(a) + \alpha_2(a) + \alpha_1(b) + \alpha_2(b) + \alpha_1(c) + \alpha_2(c) = 1. \tag{7}$$

Then it follows that the linear ODE must be of the form

$$y'' + p(x)y' + q(x)y = 0$$

for

$$p(x) = \frac{1 - \alpha_1(a) - \alpha_2(a)}{x - a} + \frac{1 - \alpha_1(b) - \alpha_2(b)}{x - b} + \frac{1 - \alpha_1(b) - \alpha_2(b)}{x - b},$$

and

$$q(x) = \frac{\alpha_1(a)\alpha_2(a)(a - b)(a - c)}{(x - a)^2(x - b)(x - c)} + \frac{\alpha_1(b)\alpha_2(b)(b - c)(b - a)}{(x - a)(x - b)^2(x - c)} + \frac{\alpha_1(b)\alpha_2(b)(c - a)(c - b)}{(x - a)(x - b)(x - c)^2},$$

such that condition (7) is satisfied. The solutions of this equation are usually denoted by the Riemann P-symbol:

$$y(x) = P \begin{Bmatrix} a & b & c & \\ \alpha_1(a) & \alpha_1(b) & \alpha_1(c) & x \\ \alpha_2(a) & \alpha_2(b) & \alpha_2(c) & \end{Bmatrix}.$$

The ordinary hypergeometric function is a special case where we choose $(a, b, c) = (0, \infty, 1)$ and pose special conditions on the α's, may be expressed as

$$
{}_2F_1(\hat{a}, \hat{b}; \hat{c}; x) = P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \hat{a} & 0 & x \\ 1-\hat{c} & \hat{b} & \hat{c}-\hat{a}-\hat{b} & \end{Bmatrix}.
$$

Now there are various identities about the P-symbol. For example, a natural question is what happens to the solution if we act on the regular singular points (a, b, c) with a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$. It turns out that denoting

$$
u = \frac{Ax+B}{Cx+D}, \quad \eta = \frac{Aa+B}{Ca+D}, \quad \zeta = \frac{Ab+B}{Cb+D}, \quad \theta = \frac{Ac+B}{Cc+D},
$$

we find

$$
P \begin{Bmatrix} a & b & c & \\ \alpha_1(a) & \alpha_1(b) & \alpha_1(c) & x \\ \alpha_2(a) & \alpha_2(b) & \alpha_2(c) & \end{Bmatrix} = P \begin{Bmatrix} \eta & \zeta & \theta & \\ \alpha_1(a) & \alpha_1(b) & \alpha_1(c) & u \\ \alpha_2(a) & \alpha_2(b) & \alpha_2(c) & \end{Bmatrix}.
$$

This is indeed useful since as $\mathrm{GL}(2, \mathbb{C})$ operates 3-transitively, we may choose $(\eta, \zeta, \theta) = (0, \infty, 1)$. The following identity fully justifies Riemann's claim that every second-order linear ODE with three regular singular points can be transformed into an equation coming from a hypergeometric function:

$$
\begin{aligned}
&P \begin{Bmatrix} a & b & c & \\ \alpha_1(a) & \alpha_1(b) & \alpha_1(c) & x \\ \alpha_2(a) & \alpha_2(b) & \alpha_2(c) & \end{Bmatrix} \\
=&C \cdot P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \alpha_1(a)+\alpha_1(b)+\alpha_1(c) & 0 & \frac{(x-a)(c-b)}{(x-b)(c-a)} \\ \alpha_2(a)-\alpha_1(a) & \alpha_1(a)+\alpha_2(b)+\alpha_1(c) & \alpha_2(c)-\alpha_1(c) & \end{Bmatrix},
\end{aligned}
$$

where

$$
C = \left(\frac{x-a}{x-b}\right)^{\alpha} \left(\frac{x-c}{x-b}\right)^{\gamma}.
$$

2.4 Algebraic Hypergeometric Functions: The List of Schwarz

We want to answer the natural question: For which triples (a, b, c) is the hypergeometric function ${}_2F_1(a, b; c; x)$ algebraic? By Theorem 3 this is equivalent to characterizing those triples (a, b, c) for which the monodromy group of the hypergeometric equation is finite, since in this case both solution of that equation will be algebraic. The list below originally appeared in [Sch73] and became known as the Schwarz's list. It works in the following way (see also in the original paper on the pages 323-324):

Given ${}_2F_1(a,b;c;x)$ a hypergeometric function denote $(\lambda,\mu,\nu) = (|1-c|, |c-a-b|, |a-b|)$. Then define λ' to be λ mod 2 if it lies in the interval $[0,1]$, otherwise define $\lambda' = 2-\lambda$ mod 2 (which is then in the interval $[0,1]$). Analogously μ' and ν' are defined. Finally choose from the four rows

$$\begin{array}{ccc} \lambda' & \mu' & \nu' \\ \lambda' & 1-\mu' & 1-\nu' \\ 1-\lambda' & \mu' & 1-\nu' \\ 1-\lambda' & 1-\mu' & \nu' \end{array}$$

the one with the smallest sum. Finally order the obtained triple in descending order and denote the resulting triple by (λ'',μ'',ν''). The following theorem applies.

Theorem 4 (Schwarz) *${}_2F_1(a,b;c;x)$ is algebraic if and only if (λ'',μ'',ν'') appears in a row below:*

N	λ''	μ''	ν''
1	1/2	1/2	p/n
2	1/2	1/3	1/3
3	2/3	1/3	1/3
4	1/2	1/3	1/4
5	2/3	1/4	1/4
6	1/2	1/3	1/5
7	2/5	1/3	1/3
8	2/3	1/5	1/5
9	1/2	2/5	1/5
10	3/5	1/3	1/5
11	2/5	2/5	2/5
12	2/3	1/3	1/5
13	4/5	1/5	1/5
14	1/2	2/5	1/3
15	3/5	2/5	1/3

Note that, since all cases above are rational numbers, we see that algebraicity of ${}_2F_1(a,b;c;x)$ implies rationality of a, b and c.

Example: Consider the hypergeometric equation with parameters $a = 1/3, b = -1/6, c = 3/2$, i.e. the function:

$${}_2F_1\left(\frac{1}{3}, -\frac{1}{6}; \frac{3}{2}, x\right) = 1 - \frac{1}{9}x - \frac{10}{243}x^2 - \frac{154}{6561}x^3 + \cdots .$$

We have $\lambda = \nu = 1/2$ and $\mu = 1/3$ and since all of those are already between 0 and 1, we have $\lambda' = \lambda$, $\mu' = \mu$ and $\nu' = \nu$. The smallest sum is archived if we leave the triple as it is, therefore we are in case 1 of Schwarz' list and the function is algebraic! Indeed, one has

$${}_2F_1\left(\frac{1}{3}, -\frac{1}{6}; \frac{3}{2}, x\right) = \frac{(1+\sqrt{x})^{1/3} + (1-\sqrt{x})^{1/3}}{2}.$$

2.5 The Circumference of an Ellipse

We now turn to a classical example involving hypergeometric functions. Consider the family of ellipses with a fixed half-axis of length 1 and a variable half-axis of length $0 < t < 1$.

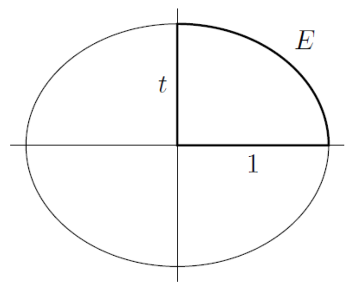

Figure 1: Ellipse of eccentricity t and quarter-length E.

The arc length of such an ellipse is given by the integral

$$\int_0^{2\pi} \sqrt{\cos^2\theta + t^2\sin^2\theta}\, d\theta = 4\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta + t^2\sin^2\theta}\, d\theta.$$

Equivalently, such an ellipse can also be described by its eccentricity $k := \sqrt{1-t^2}$. Indeed, the above integral can be rewritten as

$$\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta + t^2\sin^2\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2\sin^2\theta}\, d\theta =: E(k).$$

Our aim is to find an expression for E. Since $k^2 < 1$, this function is smooth and in fact, it is even analytic as can be seen by substituting $x = \sin\theta$ and expanding the square root in the denominator into a power series in k.

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} dx = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} k^{2n} \int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}}\, dx.$$

The integral on the right hand side is actually equal to $\frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)}$, so using the identities

$$z\Gamma(z) = \Gamma(z+1) \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma(1) = 1 \qquad \binom{n-\frac{1}{2}}{n} = (-1)^n\binom{-\frac{1}{2}}{n}$$

this fraction reduces to $\frac{\pi}{2}(-1)^n\binom{-\frac{1}{2}}{n}$ and we finally arrive at the power series expansion

$$E(k) = \sum_{n=0}^{\infty} (-1)^n \binom{\frac{1}{2}}{n} k^{2n} \frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)} = \frac{\pi}{2}\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n}\binom{-\frac{1}{2}}{n} k^{2n}.$$

Rewriting the binomial coefficients in terms of the Pochhammer symbol now shows that this series is in fact a special configuration of the hypergeometric function.

$$E(k) = \frac{\pi}{2}\,{}_2F_1\bigg(-\frac{1}{2}, \frac{1}{2}; 1; k^2\bigg).$$

Therefore, substituting $z = k^2$ implies that the function $\tilde{E}(z) := E(\sqrt{z})$ is a solution of the hypergeometric equation

$$z(1-z)\tilde{E}''(z) + (1-z)\tilde{E}'(z) + \frac{1}{4}\tilde{E}(z) = 0.$$

In terms of our original variable k, we calculate

$$\tilde{E}'(z) = \frac{d}{dz}E(\sqrt{z}) = \frac{1}{2\sqrt{z}}E'(\sqrt{z}) = \frac{1}{2k}E'(k),$$

$$\tilde{E}''(z) = \frac{d^2}{dz^2}E(\sqrt{z}) = \frac{1}{4z}E''(\sqrt{z}) - \frac{1}{4z^{\frac{3}{2}}}E'(\sqrt{z}) = \frac{1}{4k^2}E''(k) - \frac{1}{4k^3}E'(k).$$

Inserting these expressions in the hypergeometric equation above and multiplying by $-4k$ shows that E itself satisfies the second order equation

$$k(k^2-1)E''(k) + (k^2-1)E'(k) - kE(k) = 0.$$

This equation has a singularity at $k = 1$ which can be nicely interpreted. Note that the power series expansion only converges for $|k| < 1$, which is reflected in the intuitive picture since our definition of k is only reasonable for $t \in [0,1]$. However, if we had expressed E as a function of the length t of the half-axis instead, we would have expected the circumference to depend smoothly on t even when we increase it beyond $t = 1$. Analytically, this breaks down because the change of variables involves a square root, which tends to 0 as t approaches 1. Intuitively, the reason is that the definition of k required a clear distinction between the larger and shorter half-axis of the ellipse. When t passes the value 1, the roles of the two half-axes flip and an ellipse with half-axes of length 1 and $t > 1$ has the same eccentricity as an ellipse with (flipped) half-axes of length $\frac{1}{t} < 1$ and 1. So we can extend k as a function of t by the functional equation $k(t) = k(t^{-1})$ for $t > 1$, but the graph of this function has a cusp at $t = 1$.

2.6 Legendre Elliptic Curves

A more nuanced example of hypergeometric equations occurs in the context of periods of elliptic curves. An elliptic curve is said to be in Legendre normal form if its (affine) defining equation has the form $y^2 = x(x-1)(x-t)$, which defines a smooth curve as long as $t \neq 0, 1$. Every elliptic curve can be viewed as a compact Riemann surface of genus 1 and admits a nowhere-vanishing holomorphic 1-form. For a Legendre curve E_t, this can be explicitly given by

$$\omega_t = \frac{dx}{y} = \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

We can also view the Riemann surface as a two-dimensional real manifold (a torus) and integrate this form over closed curves. As a particular example, assume $0 < t < 1$ and let C_1 be the closed curve given as follows: For any $x \in \mathbb{R}$ with $x > 1$ there are two distinct values of y such that $(x, y) \in E_t$ corresponding to the positive and negative square-root of $x(x-1)(x-t)$. Going from 1 to ∞ along the curve with the positive square root and then from ∞ back to 1 along the negative then gives a closed curve in E_t. The integral becomes

$$P(t) := \int_{C_1} \omega_t = 2\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}} = 2\sum_{n=0}^\infty \binom{-\frac{1}{2}}{n}(-t)^n \int_1^\infty \frac{x^{-n-\frac{1}{2}}}{\sqrt{x(x-1)}}dx$$

$$= 2\sum_{n=0}^\infty \binom{-\frac{1}{2}}{n}(-t)^n \int_1^\infty \frac{dx}{x^{n+1}\sqrt{x-1}}$$

This integral is in fact very similar to the one in the previous example. Indeed, substituting $x = \frac{1}{y^2}$ gives

$$\int_1^\infty \frac{dx}{x^{n+1}\sqrt{x-1}} = \int_0^1 \frac{2y^{2n}}{\sqrt{1-y^2}}dy = \frac{\sqrt{\pi}\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} = \pi(-1)^n\binom{-\frac{1}{2}}{n}$$

This implies that P can also be expressed as a hypergeometric series by

$$P(t) = 2\pi\sum_{n=0}^\infty \binom{-\frac{1}{2}}{n}^2 t^n = 2\pi {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; t\right)$$

and is thus a solution of the particular hypergeometric equation

$$t(1-t)P''(t) + (1-2t)P'(t) - \frac{1}{4}P(t) = 0$$

However, we could also have chosen a different curve. If $x \in (-\infty, 0) \cup (t, 1)$, then $x(x-1)(x-t) < 0$, so the square roots are purely imaginary and can in fact be expressed as $y = \pm i\sqrt{|x|(|x|+1)(|x|+t)}$ for $x < 0$ and $\pm i\sqrt{x(1-x)(x-t)}$ for $x \in (t, 1)$. Similar to the above, we can patch the integrals and the square-root together in the right way to obtain a closed curve C_2 in E_t consisting precisely of these points. In this case, we find

$$Q(t) := \int_{C_2} \omega_t = 2\int_0^\infty \frac{dx}{i\sqrt{|x|(|x|+1)(|x|+t)}} + 2\int_t^1 \frac{dx}{i\sqrt{x(1-x)(x-t)}}$$

The exact value of the integral above can be worked out to be

$$Q(t) = \frac{2}{i}\sum_{n=0}^\infty \binom{-\frac{1}{2}}{n}^2 \left(\log\left(\frac{t}{16}\right) + 4\sum_{k=1}^n \frac{1}{n+k}\right)t^n$$

where the logarithmic term is a consequence of the domain of the second integral depending on t. Despite the more complicated expression, Q is still a solution to the differential equation above. This can be checked directly, but can also be seen from the fact that

$$t(1-t)\frac{\partial^2\omega_t}{\partial t^2} + (1-2t)\frac{\partial\omega_t}{\partial t} - \frac{1}{4}\omega_t = \frac{1}{2}d\left(\frac{x^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}{(x-t)^{\frac{1}{2}}}\right)$$

is an exact form, so by Stokes' theorem, the integral of this form over any closed curve vanishes. Since this equation has order 2 and our solutions are clearly linearly independent, the functions P and Q form a basis of the solution space at $t = 0$. Moreover, we see that the function Q is essentially a linear combination of P and the product of a logarithm with a holomorphic function. This is in fact a typical phenomenon of the hypergeometric equation in the case $c = 1$.

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